Assignment 20

Solvability by Radicals. Recapitulation.

1. Prove that the groups S_2, S_3 and S_4 are solvable.

2. Let k be a field and $n = 2d$ a positive even integer. Let $f = \sum_{j=0}^{n} a_j X^j \in k[X]$ be a monic polynomial of degree n without multiple roots and suppose that f has no root in k. Suppose moreover that f is palindromic, that is, $a_j = a_{n-j}$ for each $j \in \{0, \ldots, d\}$. Let $E = \mathrm{Sf}(f)$.

   (a) Prove that $x \mapsto \frac{1}{x}$ is a well-defined bijection of $R(f)$.

   (b) Deduce that $\mathrm{Card}(\mathrm{Gal}(E/k))$ divides $2^d d!$

3. For each of the following polynomials, determine the Galois group of its splitting field:

   (a) $X^4 + 2X^3 + X^2 + 2X + 1 \in \mathbb{Q}[X]$ [*Hint:* Exercise 2]

   (b) $X^5 + \frac{5}{4}X^4 - \frac{5}{21} \in \mathbb{Q}[X]$

   (c) $X^4 + X + 1 \in \mathbb{F}_2[X]$

   (d) $X^{81} - t \in \mathbb{F}_3(t)[X]$

4. Let k be a field.

   (a) Prove that k is an extension of a field k_0, called *prime field*, given by $k_0 = \mathbb{F}_p$ if $\mathrm{char}(k) = p > 0$ and $k_0 = \mathbb{Q}$ if $\mathrm{char}(k) = 0$.

   (b) Prove that any field homomorphism restricts to the identity on the prime fields.

5. We say that a field k is *perfect* if every algebraic field extension of k is separable.

   (a) Prove that a field k is perfect if and only if every polynomial $f \in k[X]$ is separable.

   (b) Show that fields of characteristic zero are perfect.

   (c) Suppose that $\mathrm{char}(k) = p > 0$. Prove that k is perfect if and only if the Frobenius homomorphism $\varphi : k \longrightarrow k$ sending $x \mapsto x^p$ is surjective.

   (d) Deduce that finite fields are perfect.

6. Let k be a finite field and consider a finite field extension $k(\alpha, \beta)/k$. Suppose that $k(\alpha) \cap k(\beta) = k$. Prove that $k(\alpha, \beta) = k(\alpha + \beta)$.