Assignment 21

Solvability by Radicals

1. Let be $(N, \cdot)$ and $(H, \cdot)$ be two groups and $\varphi : H \longrightarrow \mathrm{Aut}(N)$ a group homomorphism. Write $\varphi_h := \varphi(h) \in \mathrm{Aut}(N)$ for each $h \in H$. Define $G := N \rtimes_\varphi H$, the *(external) semidirect product of* N *and* H, as the set $N \times H$ with the binary operation

$$\forall n, n' \in N, \forall h, h' \in H,\ (n, h) \cdot_\varphi (n', h') = (n \cdot \varphi_h(n'), h \cdot h').$$

   (a) Check that $(N \rtimes_\varphi H, \cdot_\varphi)$ is a group.

   (b) Prove: there is a short exact sequence $1 \longrightarrow N \xrightarrow{j} N \rtimes_\varphi H \xrightarrow{\pi} H \longrightarrow 1$.

   (c) Deduce that $G = N \rtimes_\varphi H$ contains two subgroups N_0, H_0 with $N_0 \trianglelefteq G$, such that $N \cong N_0$ and $H \cong H_0$, satisfying the properties

$$\begin{cases} N_0 H_0 = G \\ N_0 \cap H_0 = \{1\}. \end{cases}$$

   Conversely, let G be a group, $H \leqslant G$ a subgroup and $N \trianglelefteq G$ a normal subgroup. We say that G is the *(inner) semidirect product of* N *and* H, if

$$\begin{cases} NH = G \\ N \cap H = \{1\}. \end{cases}$$

   In this case, we write $G = N \rtimes H$. Assume that this is the case.

   (d) Prove: there is a unique homomorphism $\alpha : G \longrightarrow H$ such that $\alpha|_H = \mathrm{id}_H$ and $\ker(\alpha) = N$.

   (e) Let $\varphi : H \longrightarrow \mathrm{Aut}(N)$ be the action of H on N by conjugation, that is, $\varphi(h)(n) := hnh^{-1}$ for all $h \in H$ and $n \in N$. Show that there is an isomorphism $\theta : G \xrightarrow{\sim} N \rtimes_\varphi H$ which satisfies $\theta|_N = j$ and $\alpha = \pi \circ \theta$. Draw a diagram containing two short exact sequences which explains the situation.

   (f) Let M be a normal subgroup of N. Show that $M \trianglelefteq G$ if and only if $hMh^{-1} = M$ for all $h \in H$.

2. Let p be a prime number and $n \geqslant 1$ an integer. Consider the natural action $\varphi : \mathrm{GL}_n(\mathbb{F}_p) \longrightarrow \mathrm{Aut}(\mathbb{F}_p^n)$. Let $G = \mathbb{F}_p^n \rtimes_\varphi \mathrm{GL}_n(\mathbb{F}_p)$ and embed $\mathbb{F}_p^n \hookrightarrow G$ via Exercise 1. Let $L \leqslant \mathbb{F}_p^n$ be a $\mathbb{F}_p$-linear subspace of $\mathbb{F}_p^n$. Show that L is subnormal in G and that $L \trianglelefteq G$ if and only if $L = 0$ or $L = \mathbb{F}_p^n$.

3. Let G be a finite group.

   (a) Suppose that G has a normal subgroup $N \trianglelefteq G$ such that G/N is abelian. Prove that G has a normal subgroup of prime index, which contains N.

   (b) Prove that G is solvable if and only if it has a normal series all whose factors are cyclic of prime order.

4. Let k be a field and $f \in k[X]$ a polynomial of prime degree p. Let $E = \mathrm{Sf}(f)$. Suppose that $\mathrm{Gal}(E/k)$ is cyclic of order p. Prove that f is irreducible.