Assignment 22

Fixed subfield

1. Let E/k be a splitting field of $X^n - 1 \in k[X]$ and $\Gamma_n(E)$ the subgroup of $E^\times$ of n-th roots of unity. Show that
   (a) If $\mathrm{char}(k) = 0$, then $|\Gamma_n(E)| = n$.
   (b) If $\mathrm{char}(k) = p$, and $n = p^\ell m$ with $p \nmid m$, then $|\Gamma_n(E)| = m$.

2. Let $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Recall that $\mathrm{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. List all subgroups of $\mathrm{Gal}(\mathbb{Q}(\sqrt{2}, \sqrt{3})$ and for each subgroup H determine the subfield E^H.

3. Let $p > 2$ be a prime number and $\zeta := e^{\frac{2\pi i}{p}}$. Let $E = \mathbb{Q}(\zeta)$. Recall that $\mathrm{Gal}(E/\mathbb{Q}) \cong (\mathbb{Z}/p\mathbb{Z})^\times$.
   (a) Show that there exists a unique subgroup H of $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ of order 2. What is its generator? [*Hint:* It is an element of order 2]
   (b) Prove that $\mathbb{Q}(\zeta + \zeta^{-1}) \subseteq E^H$ and that $[E : \mathbb{Q}(\zeta + \zeta^{-1})] \leqslant 2$.
   (c) Deduce that $E^H = \mathbb{Q}(\zeta + \zeta^{-1})$.