Def. III. 13

G is solvable if there exists a sequence

$$G = G_0 \triangleright G_1 \triangleright G_2 \cdots \triangleright G_n = (e)$$

of subgroups s.t.

(1) $G_{i+1} \triangleleft G_i \qquad 0 \leq i \leq n-1$

(2) G_i / G_{i+1} is abelian.

Remark III.14 In the context of lemma III.9 $Gal(K_t/k)$ is solvable.

The next lemma gives a kind of procedure to check when a group is solvable:

Lemma III.15 : The following are equivalent

(1) G is solvable

(2) there is $n \geq 1$ such that $G^{(n)} = (e)$.

Proof: (2) $\Rightarrow$ (1) is clear.

(1) $\Rightarrow$ (2):

Since G_i / G_{i+1} is abelian, we have

$$G_i^{(1)} \subset G_{i+1} \quad ;$$

thus $G^{(1)} \subset G_1$, $G^{(2)} = [G^{(1)}, G^{(1)}] \subset G_2$

and by recurrence $G^{(n)} \subset G_n = \{e\}$.

$\boxed{\xi}$

Examples III.16

(1) S_2, S_3, S_4 are solvable (exercise).

S_5 is not: $[S_5, S_5] = A_5$, and since

A_5 is simple non-abelian,

$$[A_5, A_5] = A_5.$$

Thus the derived series stops at A_5.

$T_n^{(1)} = [T_n, T_n] \subset U_n.$

Next observe that

$$\begin{pmatrix} 1 & u_{12} & \cdots & u_{1n} \\ & 1 & u_{23} & \\ & & \ddots & \ddots \\ 0 & & & 1 \end{pmatrix} \longrightarrow (u_{12}, u_{23}, \ldots, u_{n-1n})$$

is a homomorphism $U_n \longrightarrow k^{n-1}$

with kernel

$$U_{n,1} = \left\{ \begin{pmatrix} 1 & 0 & \cdots & * \\ & \ddots & \ddots & \\ & & \ddots & 0 \\ & & & 1 \end{pmatrix} \right\}$$

and thus $T_n^{(2)} \subset U_n^{(1)} \subset U_{n,1}$.

Continuing like this one concludes that

$$T_n^{(n)} = \{ Id_n \}$$

and T_n is solvable.

(2) Let k be any field. Then

$$T_n = \left\{ \begin{pmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} : \prod a_{ii} \neq 0 \right\}$$

is solvable.

Indeed: let

$$D_n = \left\{ \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} : \prod a_{ii} \neq 0 \right\}$$

Then one verifies that

$$T_n \longrightarrow D_n$$

$$\begin{pmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} \longmapsto \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}$$

is a homomorphism ~~and since D_n is abelian, $T_n^{(1)} = [T_n, T_n] \subset$~~.

with kernel

$$U_n = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$$

we have

For us the crucial fact will be:

Prop. III.17

(1) $H < G$, then G solvable $\Rightarrow$ H solvable.

(2) Let $N \triangleleft G$: then G is solvable iff N and $G/_N$ are solvable.

Proof:

(1) By lemma III.12 (2) applied to the injection $H \hookrightarrow G$, we have

$$H^{(k)} \subset G^{(k)} \quad \forall k \geq 0.$$

(2) Denote by $\pi : G \longrightarrow G/_N =: L$ the canonical projection. Then by the same lemma we have

$$\pi(G^{(k)}) = L^{(k)}, \quad \forall k \geq 0.$$

Thus if G is solvable, there is $n \geq 1$ with $G^{(n)} = (e)$ hence $L^{(n)} = (e)$ and $L = G/_N$ is solvable; N is solvable by (a).

Conversely, assume N and $L = G/_N$ are solvable; let $k \geq 0$ be such that $L^{(k)} = (e)$. Then $\pi(G^{(k)}) = e$ hence $G^{(k)} \subset N$. Let $j \geq 0$ such that $N^{(j)} = e$. Then $G^{(k+j)} = (G^{(k)})^{(j)}$

$\subset N^{(j)} = e$. $\boxed{\Sigma}$

Now we are in a position to show

Thm III.18. Let $f \in k[x]$ and E a splitting field of f. If f is solvable by radicals then $Gal(E/k)$ is a solvable group.

Proof:

Using Cor. III.8, let $K \supset E \supset k$ be a radical normal extension. Let

$K = K_t \supset K_{t-1} \ldots \supset K_0 = k$, be such that $K_i = K_{i-1}(u_i)$, is for $1 \leq i \leq t$, a ~~radical ex~~ pure extension of prime type p_i, that is: $u_i^{p_i} \in K_{i-1}$. Let

$$m = \prod_{i=1}^{t} p_i$$

K^* the splitting field of $X^m - 1 \in K[x]$ and k^* the splitting field of $X^m - 1 \in k[x]$.

Then, since K/k is normal and $X^m - 1$ has coefficients in k, K^*/k is normal as well.

We have then the following diagramm of field inclusions:

$$\begin{tikzcd} K^* & K \arrow[l, hook'] & k \arrow[l, hook'] \arrow[dl, hook] \\ & k^* \arrow[ul] & \end{tikzcd}$$

Now we are going to exhibit a tower of pure extensions from k^* to K^*.

Define $K_0^* = k^*$, $K_1^* = K_0^*(u_1)$, and inductively $K_i^* = K_{i-1}^*(u_i)$.

Observe that $\cancel{K_{i-1}}$ $u_i^{p_i} \in K_{i-1}^*$. ~~Indeed~~ hence $u_i^{p_i} \in K_{i-1}^*$

~~for $i = 1$: $u_1^{p_1} \in k \subset k^* = K_0^*$.~~ $\cancel{k}$

(exercise).

~~By recurrence, assuming~~

~~Pass~~

Since p_i is a prime number we have for every i two possibilities:

(1) $K_i^* = K_{i-1}^*$, that is, $u_i \in K_{i-1}^*$

(2) $K_i^* = K_{i-1}^*(u_i)$ is a pure extension of type p_i.

By lemma III.9, this implies that $Gal(K^*/k^*)$ is solvable. Observe that since K^*/k is a normal extension, Thm II.26 says that $Gal(K^*/k^*)$ is isomorphic to the quotient of $Gal(K^*/k)$ by $Gal(k^*/k)$.

Now let $\Gamma_m(k^*) = \{\xi \in k^* : \xi^m = 1\}$, be the set of roots of $X^m - 1$ in k^*.

Observe that $\Gamma_m(k^*)$ is a finite cyclic group and that every $\sigma \in Gal(k^*/k)$ gives by restriction an automorphism of $\Gamma_m(k^*)$. The homomorphism

$$Gal(k^*/k) \longrightarrow Aut\,\Gamma_m(k^*)$$

being injective and $Aut\,\Gamma_m(k^*)$ abelian implies that $Gal(k^*/k)$ is abelian. Thus by Prop. III.17 (2) $Gal(K^*/k)$ is solvable and so is $Gal(E/k)$ since it is the quotient of $Gal(K^*/k)$ by $Gal(K^*/E)$. $\blacksquare$

With Coroll. II.22 we conclude:

<u>Corollary III.19</u> The equation $X^5 - 4X + 2$ is not solvable by radicals.