D-MATH | Algebra I+II | HS17
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Midterm

1.	Let G be a group and H a subgroup of G.	true	false
a)	The index of H in G is a prime number.		
b)	If H is abelian, then H is normal in G.		
c)	If G is abelian, then H is normal in G.		
d)	If H is abelian and the index of H in G is two, then G is abelian.		
e)	If G is simple, then either $H = G$ or $H = \{1\}$.		

2.	Let G be a group acting on a set X and H a subgroup of G.	true	false
a)	If the action of G is faithful, so is the action of H on X.		
b)	If the action of G has no fixed point, so does the action of H on X.		
c)	If the action of G is transitive, so is the action of H on X.		
d)	For each $x \in X$, $\mathrm{Stab}_H(x) = \mathrm{Stab}_G(x) \cap H$.		
e)	Each H-orbit of X is contained in a G-orbit of X.		

3.		true	false
a)	A_7 is simple.		
b)	Every permutation has a unique decomposition into a product of transpositions.		
c)	One can write down $(1\,2\,3\,4\,5)$ as a product of exactly 5 transpositions.		
d)	The action of S_n on $\{1, \dots, n\}$ is transitive and faithful.		
e)	The permutations $(1\,2\,3)(4\,5)$ and $(1\,5)(2\,3\,4)$ are conjugated in S_8.		

4. Let A be a commutative ring and I an ideal of A.	true	false
a) If $f, g \in A[X]$ have both degree 3, then $f \cdot g$ has degree 6.		
b) If I is a maximal ideal, then A/I is an integral domain.		
c) If A is a UFD, then $A[X]$ is a PID.		
d) If A is a PID, then A/I is a PID.		
e) The set of polynomials in $A[X]$ whose coefficients lie in I is an ideal in $A[X]$.		

5. Let A and B be commutative rings and $f : A \longrightarrow B$ a ring homomorphism. Let I be an ideal in A and denote by $p : A \longrightarrow A/I$ the usual projection.	true	false
a) If $\ker(f) \subset I$, then there exists a ring homomorphism $g : A/I \longrightarrow B$ such that $f = g \circ p$.		
b) If B is a field, then $A/\ker(f)$ is an integral domain.		
c) If f is injective, then A is isomorphic to a subring of B.		
d) For every $b \in B$, there exists a unique ring homomorphism $h : A[X] \longrightarrow B$ sending $X \longmapsto b$.		
e) If $J \subset B$ is a prime ideal, then $f^{-1}(J)$ is a prime ideal in A.		

6. Let $R = \mathbb{Z}/15\mathbb{Z}$.	true	false
a) R contains precisely 2 prime ideals.		
b) $R[X]$ is a PID.		
c) The ideal generated by X in $R[X]$ is maximal.		
d) $\mathrm{card}(R^\times) = 8$		
e) There exists precisely one ring homomorphism $R \longrightarrow \mathbb{Z}$.		

7.	true	false
a) A free $\mathbb{Z}$-module has no torsion.		
b) A free $\mathbb{Z}$-module is finitely generated.		
c) There are, up to isomorphism, 3 different abelian groups of 18 elements.		
d) There are, up to isomorphism, 4 different abelian group of 100 elements.		
e) The $\mathbb{Z}$-module $\mathbb{Q}$ is free.		

8.	Let L/K be a field extension.	true	false
a)	If L/K is of finite degree, then it is algebraic.		
b)	If $f \in K[T]$ has no roots in L, then it is irreducible in $K[T]$.		
c)	If $\alpha \in L$ is algebraic, then $\deg(\mathrm{irr}(\alpha; K)) = [K(\alpha) : K]$.		
d)	If $\alpha, \beta \in L$ are transcendental over K, then $\alpha + \beta$ is transcendental over K.		
e)	If $\alpha \in L \setminus K$ and $\alpha^2 \in K$, then $\mathrm{irr}(\alpha; K) = X^2 - \alpha^2$.		

9.		true	false
a)	There exists a finite field with 250 elements.		
b)	Any finite field with 81 elements has characteristic 3.		
c)	The polynomial $X^{120} - 1 \in \mathbb{F}_{11}[X]$ has 10 roots in $\mathbb{F}_{11}$.		
d)	If E is a finite field and F/E is an algebraic field extension, then F is a finite field.		
e)	If E is a finite field with m elements and F/E is a finite field extension of E, then $\mathrm{card}(F)$ is a multiple of m.		

10.	Consider the polynomial $f = X^5 - 1 \in \mathbb{Q}[X]$ and let $K/\mathbb{Q}$ be the splitting field of f in $\mathbb{C}$.	true	false
a)	$K/\mathbb{Q}$ has degree divisible by 6.		
b)	f is the minimal polynomial of $e^{\frac{2\pi i}{5}}$ over $\mathbb{Q}$.		
c)	$\cos(\frac{2\pi}{5}) \in K$		
d)	Any field homomorphism $K \longrightarrow \mathbb{C}$ has image equal to K.		
e)	$\mathbb{Q}(e^{\frac{2\pi i}{5}}) = K$.		