ETH Zürich
Prof. Dr. A. Iozzi

Summer 2019
13 August

Exam Analysis III
D–MAVT, D–MATL

Please fill!

Surname:	
First Name:	
Student Card Nr.:	

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Exercise	Value	Points	Control
1	8		
2	10		
3	15		
4	6		
5	8		
Total	47		

Please do not fill!

Completeness	

Please turn!

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Important: Before the exam starts, please

- Turn off your mobile phone and place it inside your briefcase/backpack.
- Put your bags on the floor. No bags on the desk!
- Place your Student Card (Legi) on the desk.
- Fill in the front page of the exam with your generalities.

During the exam, please

- Start every exercise on a new piece of paper.
- Put your name on the top right corner of every page.
- You are expected to motivate your answers. Please write down calculations and intermediate results.
- Provide at most **one** solution to each exercise.
- **Do not** write with **pencils**. Please avoid using **red** or **green** ink pens.

Allowed aids:

- 20 pages (= 10 sheets) DIN A4 handwritten or typed personal summary.
- An English (or English-German) dictionary.

Not allowed:

No further aids are allowed. In particular neither communication devices, nor pocket calculators.

Good Luck!

See the next page!

Laplace Transforms: ($F = \mathcal{L}(f)$)

	$f(t)$	$F(s)$
1)	1	$\frac{1}{s}$
2)	t	$\frac{1}{s^2}$
3)	t^2	$\frac{2}{s^3}$
4)	$t^n,\ n \in \mathbb{Z}_{\geqslant 0}$	$\frac{n!}{s^{n+1}}$

	$f(t)$	$F(s)$
5)	$t^a,\ a > 0$	$\frac{\Gamma(a+1)}{s^{a+1}}$
6)	e^{at}	$\frac{1}{s-a}$
7)	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
8)	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$

	$f(t)$	$F(s)$
9)	$\cosh(at)$	$\frac{s}{s^2-a^2}$
10)	$\sinh(at)$	$\frac{a}{s^2-a^2}$
11)	$u(t-a)$	$\frac{1}{s}e^{-as}$
12)	$\delta(t-a)$	e^{-as}

(Γ = Gamma function, u = Heaviside function, δ = Delta function)

Remember that:

- D_R is the disk of radius $R > 0$, centered in the origin:
$D_R = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leqslant R^2\}$ and ∂D_R is its boundary:
$\partial D_R = \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = R^2\}$.

- You might need to use in some exercises the following integrals:
$$\int \sin^2(\vartheta)\, d\vartheta = \frac{1}{2}\big(\vartheta - \sin(\vartheta)\cos(\vartheta)\big) \quad \text{(+ constant)}$$
$$\int \cos^2(\vartheta)\, d\vartheta = \frac{1}{2}\big(\vartheta + \sin(\vartheta)\cos(\vartheta)\big) \quad \text{(+ constant)}$$

- For $a > 0$, the Fourier transform of the gaussian function e^{-ax^2} is
$$\widehat{e^{-ax^2}} = \frac{1}{\sqrt{2a}} e^{-\frac{\omega^2}{4a}}$$

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1. Laplace Transform (*8 Points*)

Find the solution $f : [0, +\infty) \to \mathbb{R}$ of the following integral equation:

$$f(t) = t^3 + \int_0^t e^{-(t-\tau)} f(\tau)\, d\tau. \qquad (1)$$

2. Short Questions (*10 Points*)

Answer the following questions. You can use any formula from the script.

a) (*2 Points*) The integral

$$h(x) = \frac{2}{\pi} \int_0^{+\infty} \frac{\sin(\omega)\cos(\omega x)}{\omega}\, d\omega$$

is the Fourier integral of the function

$$f(x) = \begin{cases} 1, & |x| \leqslant 1 \\ 0. & |x| > 1 \end{cases}$$

Find the explicit values of $h(x)$ for each $x \in \mathbb{R}$.

b) (*2 Points*) Consider the following PDE:

$$xu_{xx} + 2yu_{xy} + xu_{yy} = e^{-x}u + u_x.$$

In which region of the plane $(x, y) \in \mathbb{R}^2$ is it elliptic?

c) (*3 Points*) Consider the solution of the following wave equation:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R},\ t \geqslant 0 \\ u(x,0) = e^{2x}, & x \in \mathbb{R} \\ u_t(x,0) = 0. & x \in \mathbb{R} \end{cases}$$

Find the evolution in time of the point $x = 0$:

$$u(0,t) = ?$$

d) (*3 Points*) Consider the following Laplace equation on the unit disk:

$$\begin{cases} \nabla^2 u = 0, & \text{in } D_1 \\ u = x + 5y^2. & \text{on } \partial D_1 \end{cases}$$

Find the value of the solution in the center:

$$u(0,0) = ?$$

3. Wave Equation (*15 Points*)

Find the solution of following wave equation with homogeneous Neumann conditions (= the derivative u_x on the boundary is zero):

$$u = u(x,t) \quad \text{s.t.} \qquad \begin{cases} u_{tt} = c^2 u_{xx}, & x \in [0,\pi],\ t \geqslant 0 \\ u_x(0,t) = u_x(\pi,t) = 0, & t \geqslant 0 \\ u(x,0) = 1 + \cos(4x), & x \in [0,\pi] \\ u_t(x,0) = 3. & x \in [0,\pi] \end{cases} \tag{2}$$

Use the method of separation of variables, showing all the steps.

4. Laplace Equation and Maximum Principle (*6 Points*)

Consider the solution of the following Laplace problem on a disk D_R centered in the origin, of radius $R > 0$:

$$u = u(x,y) \quad \text{s.t.} \qquad \begin{cases} \nabla^2 u = 0, & \text{in } D_R \\ u = \frac{e^R}{2R^2}\left(x^2 - y^2\right). & \text{on } \partial D_R \end{cases} \tag{3}$$

Find the unique $R > 0$ such that the maximum of u on the disk is π:

$$\max_{(x,y)\in D_R} u(x,y) = \pi.$$

5. Heat Equation via Fourier Transform (*8 Points*)

Remember that the solution of the heat equation

$$\begin{cases} u_t = c^2 u_{xx}, & x \in \mathbb{R},\ t \geqslant 0 \\ u(x,0) = f(x), & x \in \mathbb{R} \end{cases}$$

has Fourier transform

$$\widehat{u}(\omega,t) = \widehat{f}(\omega)e^{-c^2\omega^2 t}.$$

For some particular cases of f, this $\widehat{u}$ can be recognised as the Fourier transform of some function, and the original solution $u = u(x,t)$ can be found.

Find the solution $u = u(x,t)$ of the following:

$$\begin{cases} u_t = c^2 u_{xx}, & t \geqslant 0,\ x \in \mathbb{R} \\ u(x,0) = e^{-\frac{1}{2}x^2}. & x \in \mathbb{R} \end{cases} \tag{4}$$

[*Hint:* The Fourier transform of this gaussian function $f(x) = e^{-\frac{1}{2}x^2}$ is recalled at the beginning of the exam.]