D-MAVT
D-MATL

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Analysis III
Exam Solutions

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Exercise	1	2	3	4	5	Total
Value	8	10	15	6	8	47

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1. Laplace Transform (*8 Points*)

Find the solution $f : [0, +\infty) \to \mathbb{R}$ of the following integral equation:

$$f(t) = t^3 + \int_0^t e^{-(t-\tau)} f(\tau)\, d\tau. \tag{1}$$

Solution:

We apply the Laplace transform to both sides. We denote the Laplace transform of f by $F = \mathcal{L}(f)$. The integral on the right-hand side is a convolution:

$$\int_0^t e^{-(t-\tau)} f(\tau)\, d\tau = \left(e^{-t} * f\right)(t).$$

The Laplace transform of a convolution is the product of the Laplace transforms of the two functions, therefore:

$$\mathcal{L}\left(e^{-t} * f\right) = \mathcal{L}(e^{-t}) \cdot \mathcal{L}(f) = \frac{1}{s+1} \cdot F(s).$$

Now we can transform the integral equation (1) into the algebraic equation

$$\begin{aligned} & F(s) = \frac{6}{s^4} + \frac{1}{s+1} \cdot F(s) \\ \implies & \left(1 - \frac{1}{s+1}\right) F(s) = \frac{6}{s^4} \\ \implies & \frac{s}{s+1} F(s) = \frac{6}{s^4} \\ \implies & F(s) = \frac{6}{s^4} + \frac{6}{s^5}. \end{aligned}$$

Finally we transform back to obtain:

$$\boxed{f(t) = \mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{6}{s^4} + \frac{6}{s^5}\right) = t^3 + \frac{t^4}{4}.}$$

2. Short Questions (*10 Points*)

Answer the following questions. You can use any formula from the script.

a) (*2 Points*) The integral

$$h(x) = \frac{2}{\pi} \int_0^{+\infty} \frac{\sin(\omega)\cos(\omega x)}{\omega}\, d\omega$$

is the Fourier integral of the function

$$f(x) = \begin{cases} 1, & |x| \leqslant 1 \\ 0. & |x| > 1 \end{cases}$$

Find the explicit values of $h(x)$ for each $x \in \mathbb{R}$.

Solution:

The Fourier integral of a function has the same value of the function itself where the function is continuous.
In a point of discontinuity x_0 the value of the Fourier integral is the average of the left and the right limit of f:

$$h(x_0) = \frac{1}{2}\left(f(x_0^+) + f(x_0^-)\right).$$

In this case the points of discontinuity are $x_0 = \pm 1$, both with average $1/2$, therefore:

$$\boxed{h(x) = \begin{cases} 1, & |x| < 1 \\ \frac{1}{2}, & |x| = 1 \\ 0. & |x| > 1 \end{cases}}$$

b) (*2 Points*) Consider the following PDE:

$$x u_{xx} + 2y u_{xy} + x u_{yy} = e^{-x} u + u_x.$$

In which region of the plane $(x, y) \in \mathbb{R}^2$ is it elliptic?

Solution:

Writing the PDE in the form

$$A u_{xx} + 2B u_{xy} + C u_{yy} = F(x, y, u, u_x, u_y),$$

the elliptic condition is $AC - B^2 > 0$. In this case this means

$$x^2 - y^2 > 0,$$

or equivalently

$$\boxed{|x| > |y|.}$$

c) (*3 Points*) Consider the solution of the following wave equation:

$$\begin{cases} u_{tt} = c^2 u_{xx}, & x \in \mathbb{R},\ t \geqslant 0 \\ u(x,0) = e^{2x}, & x \in \mathbb{R} \\ u_t(x,0) = 0. & x \in \mathbb{R} \end{cases}$$

Find the evolution in time of the point $x = 0$:

$$u(0,t) = ?$$

Solution:

We can use D'Alembert's formula:

$$u(0,t) = \frac{1}{2}\left(f(ct) + f(-ct)\right) + \int_{-ct}^{ct} \cancel{g(s)\,ds} = \frac{1}{2}\left(e^{2ct} + e^{-2ct}\right) = \boxed{\cosh(2ct).}$$

d) (*3 Points*) Consider the following Laplace equation on the unit disk:

$$\begin{cases} \nabla^2 u = 0, & \text{in } D_1 \\ u = x + 5y^2. & \text{on } \partial D_1 \end{cases}$$

Find the value of the solution in the center:

$$u(0,0) = ?$$

Solution 1:

According to the mean value principle, the value in the center is the average of the function on the boundary. We write this function in polar coordinates

$$f(\vartheta) = \cos(\vartheta) + 5\sin^2(\vartheta),$$

and we integrate it[1] to get:

$$u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} f(\vartheta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \left(\cos(\vartheta) + 5\sin^2(\vartheta)\right)\,d\vartheta =$$

$$= \frac{1}{2\pi}\left[\sin(\vartheta) + \frac{5}{2}\vartheta - \frac{5}{2}\sin(\vartheta)\cos(\vartheta)\right]\Bigg|_{\vartheta=0}^{\vartheta=2\pi} = \boxed{\frac{5}{2}.}$$

[1] We use the indefinite integral $\int \sin^2(\vartheta)\,d\vartheta = \frac{1}{2}(\vartheta - \sin(\vartheta)\cos(\vartheta)) + c$, which was given at the beginning of the exam.

Solution 2:

We can also directly solve the equation and from the solution we will read the value $u(0,0)$. This is not recommended in general because usually solving the equation is more difficult than computing an integral.
Nevertheless, here the function on the boundary is a polynomial, and the solution will also be a polynomial. One can proceed by using some "similarity method" and guessing that it is a polynomial of the form

$$u(x,y) = a + bx + cx^2 + dy^2,$$

and then $\nabla^2 u = 0$ gives $c + d = 0$, and the boundary conditions the other constants:

$$u(x,y) = \frac{5}{2} + x - \frac{5}{2}x^2 + \frac{5}{2}y^2.$$

From which $u(0,0) = 5/2$ as expected.

3. Wave Equation (*15 Points*)

Find the solution of following wave equation with homogeneous Neumann conditions (= the derivative u_x on the boundary is zero):

$$u = u(x,t) \quad \text{s.t.} \quad \begin{cases} u_{tt} = c^2 u_{xx}, & x \in [0,\pi],\ t \geqslant 0 \\ u_x(0,t) = u_x(\pi,t) = 0, & t \geqslant 0 \\ u(x,0) = 1 + \cos(4x), & x \in [0,\pi] \\ u_t(x,0) = 3. & x \in [0,\pi] \end{cases} \tag{2}$$

Use the method of separation of variables, showing all the steps.

Solution:

The separation of variables Ansatz is $u(x,t) = F(x)G(t)$ and the wave equation becomes the system of coupled equations:

$$\begin{cases} F'' = kF, \\ \ddot{G} = kc^2 G, \end{cases} \qquad \text{for some } k \in \mathbb{R}.$$

In order to have nontrivial solutions, the homogeneous Neumann conditions imply:

$$\begin{cases} u_x(0,t) = F'(0)G(t) = 0, & t \geqslant 0 \\ u_x(\pi,t) = F'(\pi)G(t) = 0, & t \geqslant 0 \end{cases} \implies \begin{cases} F'(0) = 0, \\ F'(\pi) = 0. \end{cases}$$

We have obtained a well-defined initial value problem for $F = F(x)$:

$$\begin{cases} F'' = kF, \\ F'(0) = F'(\pi) = 0. \end{cases}$$

The form of the general solution of the differential equation $F'' = kF$ depends on the positivity of k.

$k \gtreqless 0$	general solution $F(x)$	derivative $F'(x)$
$k > 0$	$Ae^{\sqrt{k}\,x} + Be^{-\sqrt{k}\,x}$	$\sqrt{k}\left(Ae^{\sqrt{k}\,x} - Be^{-\sqrt{k}\,x}\right)$
$k = 0$	$Ax + B$	B
$k < 0$	$A\cos\left(\sqrt{-k}\,x\right) + B\sin\left(\sqrt{-k}\,x\right)$	$-A\sqrt{-k}\sin\left(\sqrt{-k}\,x\right) + B\sqrt{-k}\cos\left(\sqrt{-k}\,x\right)$

The derivatives at the boundary are

$k \gtreqless 0$	$F'(0)$	$F'(\pi)$
$k > 0$	$\sqrt{k}(A - B)$	$\sqrt{k}\left(Ae^{\sqrt{k}\pi} - Be^{-\sqrt{k}\pi}\right)$
$k = 0$	B	B
$k < 0$	$B\sqrt{-k}$	$-A\sqrt{-k}\sin\left(\sqrt{-k}\,\pi\right) + B\sqrt{-k}\cos\left(\sqrt{-k}\,\pi\right)$

By imposing them equal to zero we finally get the following solutions:

$k \gtreqless 0$	nontrivial solutions?	general solution F(x)
$k > 0$	no	/
$k = 0$	yes	$F_0(x) = B$
$k < 0$	yes, for $k = -n^2$	$F_n(x) = B_n \cos(nx)$

Now for these values of k ($k = 0$ and $k = -n^2$) we solve the problem for $G = G(t)$.

k	solution $G(t)$
$k = 0$	$G_0(t) = Ct + D$
$k = -n^2$	$G_n(t) = C_n \cos(cnt) + D_n \sin(cnt)$

By the superposition principle, any function of the form

$$u(x,t) = F_0(x)G_0(t) + \sum_{n=1}^{+\infty} F_n(x)G_n(t)$$

will be a solution. By renaming the constants differently this is:

$$u(x,t) = A_0 t + B_0 + \sum_{n=1}^{+\infty} (A_n \cos(cnt) + B_n \sin(cnt)) \cos(nx).$$

The time derivative is

$$u_t(x,t) = A_0 + \sum_{n=1}^{+\infty} cn(-A_n \sin(cnt) + B_n \cos(cnt)) \cos(nx).$$

Now we have to find these constants by imposing the initial conditions

$$\begin{cases} u(x,0) = 1 + \cos(4x), \\ u_t(x,0) = 3. \end{cases}$$

In this case they translate into

$$\begin{cases} B_0 + \sum_{n=1}^{+\infty} A_n \cos(nx) = 1 + \cos(4x), \\ A_0 + \sum_{n=1}^{+\infty} cnB_n \cos(nx) = 3, \end{cases}$$

which means that the only nonzero coefficients are

$$\begin{cases} B_0 = 1, \\ A_4 = 1, \\ A_0 = 3. \end{cases}$$

Therefore the final solution is:

$$\boxed{u(x,t) = 3t + 1 + \cos(4ct)\cos(4x).}$$

4. Laplace Equation and Maximum Principle (*6 Points*)

Consider the solution of the following Laplace problem on a disk D_R centered in the origin, of radius $R > 0$:

$$u = u(x, y) \quad \text{s.t.} \qquad \begin{cases} \nabla^2 u = 0, & \text{in } D_R \\ u = \frac{e^R}{2R^2}\left(x^2 - y^2\right). & \text{on } \partial D_R \end{cases} \tag{3}$$

Find the unique $R > 0$ such that the maximum of u on the disk is π:

$$\max_{(x,y)\in D_R} u(x, y) = \pi.$$

Solution:

We find the maximum of u. This will be a function of R, which we are going to impose equal to π.
By the maximum principle we have to find the maximum of the function on the boundary

$$\max_{(x,y)\in D_R} u(x, y) = \max_{(x,y)\in \partial D_R} u(x, y) = \max_{\{x^2+y^2=R^2\}} \frac{e^R}{2R^2}\left(x^2 - y^2\right).$$

In other words we have to find the maximum of $x^2 - y^2$ in the region $\{x^2 + y^2 = R^2\}$.
To do this we can parametrise the boundary with polar coordinates and find the maximum of $R^2(\cos^2(\vartheta) - \sin^2(\vartheta)) = R^2\cos(2\vartheta)$ in the interval $\vartheta \in [0, 2\pi]$, which is clearly R^2.
An equally valid argument would be to observe that on the boundary

$$x^2 - y^2 = 2x^2 - R^2 \qquad \Longrightarrow \qquad \max\left(x^2 - y^2\right) = R^2.$$

Anyway, the maximum of u will be:

$$\max_{(x,y)\in D_R} u(x, y) = \max_{\{x^2+y^2=R^2\}} \frac{e^R}{2R^2}\left(x^2 - y^2\right) = \frac{e^R}{2\cancel{R^2}} \cdot \cancel{R^2} = \frac{e^R}{2}.$$

This is equal to π if and only if

$$\frac{e^R}{2} = \pi \quad \Leftrightarrow \quad e^R = 2\pi \quad \Leftrightarrow \quad \boxed{R = \ln(2\pi).}$$

5. Heat Equation via Fourier Transform (*8 Points*)

Remember that the solution of the heat equation

$$\begin{cases} u_t = c^2 u_{xx}, & x \in \mathbb{R},\ t \geqslant 0 \\ u(x,0) = f(x), & x \in \mathbb{R} \end{cases}$$

has Fourier transform

$$\widehat{u}(\omega, t) = \widehat{f}(\omega) e^{-c^2\omega^2 t}.$$

For some particular cases of f, this $\widehat{u}$ can be recognised as the Fourier transform of some function, and the original solution $u = u(x,t)$ can be found.

Find the solution $u = u(x,t)$ of the following:

$$\begin{cases} u_t = c^2 u_{xx}, & t \geqslant 0,\ x \in \mathbb{R} \\ u(x,0) = e^{-\frac{1}{2}x^2}. & x \in \mathbb{R} \end{cases} \tag{4}$$

[*Hint:* The Fourier transform of this gaussian function $f(x) = e^{-\frac{1}{2}x^2}$ is recalled at the beginning of the exam.]

Solution:

The Fourier transform we need is recalled at the beginning of the exam

$$\widehat{e^{-ax^2}} = \frac{1}{\sqrt{2a}} e^{-\frac{\omega^2}{4a}}. \tag{5}$$

Here $a = 1/2$ and therefore

$$\widehat{e^{-\frac{1}{2}x^2}} = e^{-\frac{\omega^2}{2}}.$$

So the solution has Fourier transform

$$\widehat{u}(\omega, t) = e^{-\frac{\omega^2}{2}} e^{-c^2\omega^2 t}.$$

We want to recognise it as a Fourier transform of some function. We notice it is still a gaussian

$$\widehat{u}(\omega, t) = e^{-\left(\frac{1}{2} + c^2 t\right)\omega^2},$$

and, because of (5), we would like to have for some $a > 0$,

$$\frac{1}{2} + c^2 t = \frac{1}{4a},$$

which gives us

$$a = \frac{1}{2 + 4c^2 t}. \tag{6}$$

We would like to say that $\widehat{u}(\omega, t)$ is the Fourier transform of the gaussian with this a, but we would miss the factor $1/\sqrt{2a}$ appearing in (5).
But then we just need to multiply and divide by it to get[2]

$$u(x,t) = \mathcal{F}^{-1}\left(\widehat{u}(\omega,t)\right) = \mathcal{F}^{-1}\left(e^{-\left(\frac{1}{2}+c^2t\right)\omega^2}\right) = \sqrt{2a}\,\mathcal{F}^{-1}\left(\frac{1}{\sqrt{2a}}e^{-\frac{\omega^2}{4a}}\right) \overset{(5)}{=}$$

$$\overset{(5)}{=} \sqrt{2a}\,e^{-ax^2} \overset{(6)}{=} \sqrt{\frac{1}{1+2c^2t}}\,e^{-\frac{x^2}{2+4c^2t}} = \boxed{\frac{e^{-\frac{x^2}{2+4c^2t}}}{\sqrt{1+2c^2t}}}.$$

An alternative way of proceeding, starting from

$$\widehat{u}(\omega,t) = e^{-\left(\frac{1}{2}+c^2t\right)\omega^2},$$

would be to use directly Fourier inversion's formula:

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\widehat{u}(\omega,t)e^{i\omega x}\,d\omega = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-\left(\frac{1}{2}+c^2t\right)\omega^2}e^{i\omega x}\,d\omega =$$

$$= \mathcal{F}\left(e^{-\left(\frac{1}{2}+c^2t\right)\omega^2}\right)(-x) \overset{(5)}{=} \frac{1}{\sqrt{2\left(\frac{1}{2}+c^2t\right)}}e^{-\frac{(-x)^2}{4\left(\frac{1}{2}+c^2t\right)}} = \frac{e^{-\frac{x^2}{2+4c^2t}}}{\sqrt{1+2c^2t}}.$$

[2]In what follows $\mathcal{F}$ denotes the Fourier transform.