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Mean-Field Games

Reading Group and Seminar

Mean-field games provide a tractable model of large population strategic games. Introduced in the seminal works of Lasry and Lions [26, 27, 28] as well as [7], and Huang, Malhamé and Caines [21], mean-field games enjoy growing interest by researchers and a wide variety of applications.

In this reading group and seminar we want to acquaint ourselves with the basic problem formulation and some approaches to study such games. The seminar will begin by recalling classical control theory, known results from stochastic differential games and then quickly progress to research-level topics on mean-field games. Over the course of the seminar we aim to gain a solid overview of current research on this topic. The typical participant will be an advanced graduate student, doctoral student or a postdoc, not necessarily an expert on the topic but interested in jointly learning the subject.

The mean-field games literature employs an interesting mix of techniques form PDEs, stochastic analysis and optimal transport theory. The seminar provides a good staring point for students to write a M.Sc. thesis in this or related areas.

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1 Outline (Preliminary)

1.1 Stochastic control primer (1)

Aim of this talk will be to introduce the basics of the stochastic control problem in its strong and weak formulation, as well as comment on the differences of open and closed loop controls. Upon introducing the basic formulation we want to understand characterizations of the value function - provided it exists - in terms of the associated Hamilton-Jacobi-Bellman equation.

References

1. Introduce the canonical stochastic control problem in its strong and weak formulation. A good reference is [8, Sec. 3.1, 3.3]. You may also find [32, Sec. 3.2], [36, Sec. 2.4.2] or [16, Ch. IV.2] helpful.

2. Discuss the Dynamic Programming Principle, *cf.* [8, Thm. 3.14]

3. Briefly discuss the HJB-equation and the verification theorem as in [8, Thm. 3.15]. Please also consult [32, Sec. 3.2-3.5] and [32, Thm. 3.5.2]. The reference [16, Ch. IV.2-IV.3] provides more background.

Prerequisites Stochastic analysis background as covered in the lecture 'Brownian Motion and Stochastic Calculus' is necessary. Please harmonize the notation across the references to give a coherent talk.

1.2 Stochastic control primer (2)

Aim of this talk will be to recall the basics of the stochastic control problem in its weak formulation and provide a BSDE characterization of the value function. This view leads to the Martingale Principle of Optimality, and gives a systematic toolbox to handle non-Markovian control problems.

1. Emphasize the BSDE formulation for the case of uncontrolled, non-degenerate diffusion-coefficients as in [8, Sec. 4.1.1, Prop. 4.1]. Please discuss the basics of BSDE and the comparison principle, *cf.* [8].
2. For the BSDE formulation [8, Sec. 2, 4] and [32, Sec. 6.4.2, Thm. 6.4.6] may also provide useful background.
3. Please discuss the comparison principle [8, Thm 2.4] and prove it if time permits.
4. A concise discussion of stochastic control via BSDE may be found in [20, Sec. II].

Prerequisites Stochastic analysis background as covered in the lecture 'Brownian Motion and Stochastic Calculus' is necessary. Familiarity with BSDEs is helpful but not essential.

1.3 Appetizer: n-player games, Nash-equilibria and stochastic differential games

The aim of this talk is to recall the basic terminology from classical strategic n-player games, define the notion of Nash-equilibrium and prove its existence. The second part of the talk should cover stochastic differential games and the generalized Isaacs condition.

References

1. Page 3 and Page 45 of [29]. Prove the exisence of classical Nash equilibria. For this please recall the relevant fixed-point theorem along the way.
2. For Nash equilibria of stochastic differential games, discuss [14, Ch. 29]. Using the comparison principle from the first talk, prove the sufficiency of the generalized Isaacs condition for the existence of Nash equilibria in stochastic differential games.
3. Time permitting, comment on similar conditions arising in e.g. [3, Eqn. 2.21] or [17, Eqn. 2.10].
4. For more background on stochastic differential games you may also find [8, Ch. 5] helpful.

Prerequisites Stochastic control theory and BSDE basics will be required for the second half. The relevant BSDE terminology can be learned from [20].

1.4 McKean-Vlasov SDE as limits of interacting particle systems

In this talk we want to understand how to obtain solutions to McKean-Vlasov type SDEs

$$dX_t = b_t(X_t, \mathrm{Law}(X_t))dt + \sigma_t(X_t, \mathrm{Law}(X_t))dB_t$$

and in which sense this equation can be viewed as a limit of a particle system.

References

1. An efficient introduction to the Wasserstein metric can be found in [33, Sec. 5.1, 5.2]. We will only need the basic definition.
2. Discuss [25], Section 3 of with emphasis Sec. 3.2, Theorem 3.3 (first part). Please cover the finite interacting particle system and the existence of a solution to the McKean-Vlasov SDE in the Lipschitz case.
3. Comment on convergence of the finite random system to the McKean-Vlasov SDE: Use the Glivenko-Cantelli Theorem, cf. [31, Thm. 7.1] to support the intuition that the random empirical measure converges to the deterministic law of X in an appropriate sense.
4. More background can be found in [8, Sec. 1.3]. A classical reference for interacting particle systems inspiring McKean-Vlasov type equations is [35].

Prerequisites Basic stochastic analysis and a firm understanding of the material from the lecture 'probability theory' is necessary.

1.5 A weak mean-field game formulation

Having recalled the basic notions from control theory and McKean-Vlasov SDEs we want to introduce a simple mean-field game formulation and discuss some examples. We begin with the concept of Nash-equilibria in a controlled diffusion environment. This talk should connect the previous talks with the mean-field game intuition that we will want to develop in the rest of the seminar. We want to introduce a weak mean-field formulation and the appropriately adapted notion of equilibrium. The talk should comment on why this notion of equilibrium makes sense and how it differs from the classical Nash concept.

References

1. Begin with the problem formulation of [3] in their introduction, then discuss [3, Sec. 1] and [3, Thm. 1.1]. Please simplify their notation and remove the discounting.
2. Proceed to the weak mean-field formulation, [23] Sections 1-3. Some remarks on the literature, the formulation, the notion of equilibrium, etc.
3. Time permitting, discuss the flocking model of [12] Section 2.2 (with original references).

Prerequisites Basic stochastic analysis and probability. Familiarity with the martingale problem formulation of SDE. Background on stochastic control is helpful, but can be easily read up on, see references above.

1.6 Interlude: The Kolmogorov-Fokker-Planck equation and PDE view

We want to familiarize ourselves with the PDE formulation of MFG. For this, we must first understand the solution $t \mapsto \mu_t$ of the Kolmogorov-Fokker-Planck equation

$$\partial_t \mu_t = \frac{1}{2}\Delta(\sigma_t(x, \mu_t)\mu_t) + \operatorname{div}(b_t(x, \mu_t)\mu_t),$$

as a flow in the space of probability measures. We then want to familiarize ourselves with the meaning of the mean-field PDE system and the Lasry-Lions uniqueness condition for equilibrium.

References

1. Revisit the McKean-Vlasov SDE, derive the Fokker-Planck equation and discuss the definition of its solution in the distributional sense. Follow [25, Sec. 3.3-3.4], you may also consult [6, Sec. 4.2.1].
2. Give the reduced mean-field PDE system [6, Sec. 4.2]. Please explicitly discuss the form of the mean-field game it represents and how it differs from the one given in the previous talk, also compare this to [6, Sec. 4.1]
3. Discuss the Lasry-Lions uniqueness condition and its proof assuming the existence of a classical solution to the PDE system. For this use [6, Sec. 4.2.3] or [18, Sec. 2.3.1].
4. A good reference for more background is the exposition [18]. Original research articles include [28, 26].

Prerequisites Basic functional analysis is relevant to understand the solution of the Fokker-Planck equation in the distributional sense. A student with strong PDE background is encouraged to take the initiative and comment more on the mean-field PDE system following the discussion in [18].

1.7 The probabilistic view (1)

We want to acquaint ourselves with the use of probabilistic techniques for MFG. The first talk should give an overview of the techniques of proof in the uncontrolled diffusion case and the weak formulation of mean-field games.

References

1. Following [12], recall the formulation of the MFG, state the important technical assumptions and the definition of equilibrium.
2. Give the main notation and discuss the technical Lemmata 7.10, 7.11 (without proof), then sketch the proof of Theorem 3.5 on page 1221 (and discuss how the regularity in the Lemmata is used in it).

Prerequisites The talk requires firm understanding of real and stochastic analysis. The relevant concepts from analysis in metric spaces are recalled in the paper and can be learned on the fly.

1.8 The probabilistic view (2)

We want to study limiting behavior of finite-player mean-field games to the canonical formulation given in the previous talk.

References Follow [12, Sec. 4]. State the main theorems and sketch the proof. The reference [24] provides more material on limiting issues.

1.9 The probabilistic view (3)

We want to discuss a more general existence proof. It allows for controlled diffusions. The proof is inspired by the classical Nash-equilibrium existence proof and uses the martingale problem formulation.

References Give the main steps of [23], also discuss the martingale measure approach of [15].

Prerequisites A firm foundation in stochastic analysis is required. The student should be familiar with the standard weak solution concept of SDE and the martingale problem formulation. Details can be read up in [34] or [22].

1.10 Interlude: Large population games without symmetry

The mean-field approach assumes symmetry and exchangeability. An approach to model large population games with players that have individual characteristics is given in [4, 5]. This analysis is inspired by transport theory. In this talk we want to gain an overview of this alternative approach, which is not phrazed as a stochastic differential game.

References Discuss the model and main results of [4] and [5]. A dynamic version of this model is found in [1]. The selection of material should be done in a way to give a coherent overview of the topic.

Prerequisites This talk does not require background in stochastic analysis, but a firm understanding of probability theory is necessary. Students that are acquainted with the basics from transportation theory as covered in the course 'optimal transport' will possess ample background to present this topic.

1.11 Mean-field games with common noise (1)

Mean-field games with common noise pose significant technical challenges. If each player in the population has independent noise, we expect cancellation in the limit. Indeed, the Kolmogorov-Fokker-Planck equation encountered in previous talks is a deterministic flow in the space of measures. In the common noise regime, this is no longer true and the mean-field system remains random in the limit. This makes the application of classical fixed-point techniques used in the independent noise regime challenging.

References Discuss the problem formulation [11] in its weak and strong form, as well as the notion of equilibrium. Comment on the technical difficulties arising in the limit if randomness persists. Conditional McKean-Vlasov dynamics can be found in [8, Sec. 1.4]. They serve as a good starting point to understand the challenges faced in the common noise regime. The books [9] [10] provides more background on the common noise regime.

1.12 Mean-field games with common noise (2)

This talk should continue the introduction to mean-field games with common noise, discussing techniques to approach the random system.

References Discuss a selection of material from [11] and [10].

1.13 Advanced Topics or Applications

TBD: Perhaps details of the FBSDE formulation (strong MFG formulation), details on the master equation, or an application.

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2 Further Topics and Application

Interesting recent applications of the mean-field modeling philosophy which do not fit into the classical framework of Lasry and Lions include:

1. Large tournament games [2]
2. Mean field optimal stopping games [30].
3. DPP for controlled McKean-Vlasov dynamics [13].
4. Machine learning algorithms for extended mean field games [19].

References

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