Probability and Statistics

Exercise sheet 1

Exercise 1.1 We throw simultaneously two dices, one green and one red. Consider the following events:

- $W_1 :=$ Neither of the dices has a result strictly greater than 2.
- $W_2 :=$ The green and the red one have the same number on them.
- $W_3 :=$ The number on the green is 3 times the number on the red.
- $W_4 :=$ The number on the red is by one greater than the number on the green one.
- $W_5 :=$ The number of the green one is greater or equal than the number on the red one.

(a) Write a suitable space Ω where all of these events can be defined.

(b) Describe W_i as a subsets of Ω.

(c) If you were colorblind (you cannot differentiate green and red). How does the sample space Ω change? Which W_i can be defined in this space?

Exercise 1.2 You have an urn with $4k$ balls. Each of these balls is numbered with a different number in $\{1, ..., 4k\}$. At time j you take out one ball, look at its number and put it back. You repeat this experiment n times. Define

- $A_j :=$ The number taken out in the j-th time is bigger than $2k$.
- $B_j :=$ The number taken out in the j-th time is even.

(a) Write in terms of $(A_j)_{j=1}^n$ and $(B_j)_{j=1}^n$ the following events

1. $A :=$ Between 1 and n there was never a number bigger than $2k$.
2. $B :=$ Between 1 and n there was at least one even number.
3. $C :=$ The amount of balls bigger than $2k$ is bigger or equal than the amount of even balls.

(b) Describe in words the following events

1. $\left(\bigcup_{j=1}^{n}(A_j)^c\right)^c$.
2. $\bigcup_{j=1}^{n-2}(A_j \cap A_{j+1} \cap B_{j+2})$.
3. $\bigcup_{m=1}^n \bigcap_{j=m}^n (A_j \cap B_j)$.

Exercise 1.3 Let $(A_j)_{j=1}^n$ be events, $A_j \subset \Omega$ and for every event A, let $\mathbb{1}_A$ denote the indicator function of A, which is a function from Ω to $\{0, 1\}$ such that $\mathbb{1}_A(w) = 1$ if $w \in A$, and $\mathbb{1}_A(w) = 0$ otherwise.

(a) Show that:

$$\mathbb{1}_{\bigcup_{j=1}^{n} A_j} = 1 - \prod_{j=1}^{n}(1 - \mathbb{1}_{A_j}),$$

use it to prove that:

$$\mathbb{P}\left[\bigcup_{j=1}^{n} A_j\right] = \sum_{k=1}^{n}(-1)^{k+1} \sum_{1\leq i_1<\ldots<i_k\leq n} \mathbb{P}\left[\bigcap_{j=1}^{k} A_{i_j}\right].$$

(b) Using induction prove the following statements:

$$\mathbb{P}\left[\bigcup_{j=1}^{n} A_j\right] \leq \sum_{j=1}^{n}\mathbb{P}[A_j] - \sum_{j=1}^{n-1}\mathbb{P}\left[A_j \cap A_{j+1}\right],$$

$$\mathbb{P}\left[\bigcup_{j=1}^{n} A_j\right] \geq \sum_{j=1}^{n}\mathbb{P}[A_j] - \sum_{i,j=1,i<j}^{n}\mathbb{P}\left[A_j \cap A_i\right].$$

Exercise 1.4 Show the Multiplication Rule for Conditional Probabilities:
Suppose that $A_1, A_2, \ldots, A_n$ are events such that $\mathbb{P}(A_1 \cap \cdots \cap A_{n-1}) > 0$. Then

$$\mathbb{P}(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\cdots\mathbb{P}(A_n|A_1 \cap \cdots \cap A_{n-1}).$$