Probability and Statistics

Exercise sheet 10

Exercise 10.1 Let $c > 0$ and consider the loss function

$$L(\theta, a) = \begin{cases} c|\theta - a| & \text{if } \theta < a, \\ |\theta - a| & \text{if } \theta \geq a. \end{cases}$$

Assume that θ is a random variable with a continuous distribution.

(a) Let $a \leq q$ be two real numbers, show that

$$\mathbb{E}[L(\theta, a) - L(\theta, q)] \geq (q - a)\left(\mathbb{P}[\theta \geq q] - c\mathbb{P}[\theta \leq q]\right),$$

and

$$\mathbb{E}[L(\theta, a) - L(\theta, q)] \leq (q - a)\left(\mathbb{P}(a \leq \theta) - c\mathbb{P}(\theta \leq a)\right).$$

(b) Prove that a Bayes estimator of θ will be any $1/(1 + c)$ quantile of the posterior distribution of θ.

Exercise 10.2 Suppose that the number of defects in a 1200-foot roll of magnetic recording tape has a Poisson distribution for which the value of the mean θ is unknown, and the prior distribution of θ is the gamma distribution with parameters $\alpha = 3$ and $\beta = 1$. When five rolls of this tape are selected at random and inspected, the numbers of defects found on the rolls are $2, 2, 6, 0$ and 3.

(a) What is the posterior distribution of θ?

(b) If the squared error loss function is used, what is the Bayes estimate of θ?

Exercise 10.3 Let $\alpha, \beta > 0$, the probability density function of a beta distribution with parameters α and β is

$$f(x|\alpha, \beta) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1 - x)^{\beta-1} & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $X_1, \cdots, X_n$ form a random sample from the Bernoulli distribution with parameter θ, which is unknown ($0 < \theta < 1$). Suppose also that the prior distribution of θ is the beta distribution with parameters $\alpha > 0$ and $\beta > 0$. Show that the posterior distribution of θ given that $X_i = x_i$, for $i = 1, \ldots, n$ is the beta distribution with parameters $\alpha + \sum_{i=1}^n x_i$ and $\beta + n - \sum_{i=1}^n x_i$.

In particular the family of beta distributions is a conjugate family of prior distributions for samples from a Bernoulli distribution. If the prior distribution of θ is a beta distribution, then the posterior distribution at each stage of sampling will also be a beta distribution, regardless of the observed values in the sample.

Exercise 10.4 Let $\xi(\theta)$ be defined as follows: for constants $\alpha > 0$ and $\beta > 0$,

$$\xi(\theta) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-(\alpha+1)} e^{-\beta/\theta} & \text{for } \theta > 0, \\ 0 & \text{for } \theta \leq 0. \end{cases}$$

A distribution with this probability density function is called an *inverse gamma distribution.*

(a) Verify that $\xi(\theta)$ is actually a probability density function.

(b) Consider the family of probability distributions that can be represented by a probability density function $\xi(\theta)$ having the given form for all possible pairs of constants $\alpha > 0$ and $\beta > 0$. Show that this family is a conjugate family of prior distributions for samples from a normal distribution with a known value of the mean μ and an unknown value of the variance θ.

Exercise 10.5 When the motion of a microscopic particle in a liquid or a gas is observed, it is seen that the motion is irregular because the particle collides frequently with other particles. The probability model for this motion, which is called *Brownian motion*, is as follows: A coordinate system is chosen in the liquid or gas. Suppose that the particle is at the origin of this coordinate system at time $t = 0$, and let (X, Y, Z) denote the coordinates of the particle at any time $t > 0$. The random variables X, Y, and Z are i.i.d., and each of them has the normal distribution with mean 0 and variance $\sigma^2 t$. Find the probability that at time $t = 2$ the particle will lie within a sphere whose center is at the origin and whose radius is 4σ.

Exercise 10.6 Take $x \in [0, 1]$. We say that x is normal if for x in its binary form,

$$x = \sum_{n \in \mathbb{N}} x_n 2^{-n} \qquad x_n \in \{0, 1\},$$

we have that $\lim_{n\to\infty} \frac{|\{1 \le k \le n : x_k = 1\}|}{n} = \frac{1}{2}$.

(a) Prove that if we have a sequence $(U_n)_{n\in\mathbb{N}}$ i.i.d. Bernoulli with parameter $\frac{1}{2}$, then $U = \sum_{n\in\mathbb{N}} U_n 2^{-n}$ is an uniform random variable in $[0, 1]$

(b) Prove that if $U \sim U(0, 1)$, $\mathbb{P}[U \text{ is normal}] = 1$.