Probability and Statistics

Exercise sheet 12

Exercise 12.1 Moore and McCabe (1999) describe an experiment conducted in Australia to study the relationship between taste and the chemical composition of cheese. One chemical whose concentration can affect taste is lactic acid. Cheese manufacturers who want to establish a loyal customer purchases the cheese. The variation in concentrations of chemicals like lactic acid can lead to variation in the taste of cheese. Suppose that we model the concentration of lactic acid in several chunks of cheese are independent normal random variables with mean μ and variance $\sigma^2 = 0.09$.

Suppose that we sample 20 chunks of cheese. Let

$$T = \sum_{i=1}^{20} (X_i - \mu)^2/20,$$

where X_i is the concentration of lactic acid in ith chunk. What number c satisfies $\mathbb{P}[T \leq c] = 0.95$?

Exercise 12.2 We toss 100 times a coin and we get 60 head. We want to do a test to know whether the coin is fair.

(a) Test the hypothesis with a 0.01 level of significance. Should this test be one or two-tailed?.

(b) What is the largest amount of heads we should have in 100 tosses so that we cannot discard $H_0 :=$ "The coin biased towards tail".

(c) Calculate all p_0 so that the null hypothesis

$$H_0(p_0) := \text{"Probability of head is } p_0 \text{ "},$$

would not be rejected in a test with 0.05 level of significance.
Hint: It will be useful to use the central limit theorem in this exercise.

Exercise 12.3 Let X be a normal random variable with $E[X] = m$ and $\mathrm{Var}(X) = \sigma^2 = 0.0014^2$. Let also X_i for $i = 1, \dots, n$ be independent random variables that share the same distribution with X. The following 12 realisations x_i of the random variables X_i were recorded:

1.00781 1.00646 1.00801 1.00833 1.00738 1.00687
1.00783 1.00936 1.00564 1.00543 1.00794 1.01060

(a) Perform a statistical test at a $\alpha = 5\%$ level of confidence for the null hypothesis $H_0 : \ \mu = 1.0085$, against the alternative hypothesis $H_A : \ \mu = 1.008$.

(b) Calculate the power of the test from part **a)**.

(c) What happens to the power calculated in part **b)** when the alternative hypothesis is changed to $H_A \ : \ \mu = 1.007$?

Exercise 12.4

When testing for bacteriological contamination of milk, $0.01\,ml$ of milk is distributed over a glass surface of $1\,cm^2$. This area is divided into 400 equal-sized squares, and for each square the number of bacterial colonies inside the square is counted with the help of a microscope. Let us assume that the following numbers N_k of squares with exactly k colonies are observed:

k	0	1	2	3	4	5	6	7	8	9	10	19
N_k	56	104	80	62	42	27	9	9	5	3	2	1

We assume that the number of bacterial colonies contained in any square, denoted by X, has a Poisson distribution, with unknown parameter λ.

(a) Find a "natural" estimator of the parameter λ of the Poisson distribution, and calculate the expected number of squares with k bacterial colonies, i.e. $400 \cdot \mathbb{P}(X = k)$.

(b) Consider the deviation of N_k from its expected value. Do you find that these fluctuations can be considered as random, or do you think that a Poisson distribution doesn't fit here?

To do further analysis, we perform a goodness of fit test:

(c) Calculate for the counts above the quantities

$$\frac{(\text{Observed count} \ - \ \text{Expected count})^2}{\text{Expected count}}$$

for all k, and the test statistic $T_N = \sum \frac{(\text{Obs.}-\text{Exp.})^2}{\text{Exp.}}$. Use the group defined by $\{k \geq 8\}$ for the tail of the distribution.

(d) Under the assumption that the error are normally distributed, and that the statistic T_N has approximately a $\chi^2(8)$ (with 8 degrees of freedom), build a test to determine if the Poisson distribution fits the data. What is the conclusion of the test for the levels of confidence $\alpha = 5\%$ and 1%?

Exercise 12.5 *Optional* Let δ be a test procedure and S_1 the critical region of δ, Θ the parameter space. The function $\pi(\theta|\delta)$, called the *power function* of the test δ, is determined by the relation

$$\pi(\theta|\delta) = \mathbb{P}(X \in S_1|\theta), \quad \text{for } \theta \in \Theta.$$

Suppose that $X_1, \cdots, X_n$ form a random sample from the normal distribution with unknown mean μ and known variance 1, and it is desired to test the following hypotheses:

$$\begin{aligned} H_0 : &\quad 0.1 \leq \mu \leq 0.2 \\ H_1 : &\quad \mu < 0.1 \text{ or } \mu > 0.2. \end{aligned}$$

Consider a test procedure δ such that the hypothesis H_0 is rejected if either $\overline{X_n} \leq c_1$ or $\overline{X_n} \geq c_2$, and let $\pi(\mu|\delta)$ denote the power function of δ. Suppose that the sample size is $n = 25$. Determine the values of the constants c_1 and c_2 such that

$$\pi(0.1|\delta) = \pi(0.2|\delta) = 0.07.$$

Standard normal (cumulative) distribution function.

$$\mathbb{P}(X \leq x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) \mathrm{d}y, \quad \text{for } x \geqslant 0$$

	0	1	2	3	4	5	6	7	8	9
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6408	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
2.2	.98610	.98645	.98679	.98713	.98745	.98778	.98809	.98840	.98870	.98899
2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.99430	.99446	.99461	.99477	.99492	.99506	.99520
2.6	.99534	.99547	.99560	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.99720	.99728	.99736
2.8	.99744	.99752	.99760	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3.0	.998650	.998694	.998736	.998777	.998817	.998856	.998893	.998930	.998965	.998999
3.1	.999032	.999065	.999096	.999126	.999155	.999184	.999211	.999238	.999264	.999289
3.2	.999313	.999336	.999359	.999381	.999402	.999423	.999443	.999462	.999481	.999499
3.3	.999517	.999534	.999550	.999566	.999581	.999596	.999610	.999624	.999638	.999651
3.4	.999663	.999675	.999687	.999698	.999709	.999720	.999730	.999740	.999749	.999758
3.5	.999767	.999776	.999784	.999792	.999800	.999807	.999815	.999822	.999828	.999835
3.6	.999841	.999847	.999853	.999858	.999864	.999869	.999874	.999879	.999883	.999888
3.7	.999892	.999896	.999900	.999904	.999908	.999912	.999915	.999918	.999922	.999925
3.8	.999928	.999931	.999933	.999936	.999938	.999941	.999943	.999946	.999948	.999950
3.9	.999952	.999954	.999956	.999958	.999959	.999961	.999963	.999964	.999966	.999967

The chi-square distribution

$$\Pr(X \le x) = \int_0^x \frac{1}{\Gamma(r/2)2^{r/2}}\, y^{r/2-1} e^{-y/2}\, \mathrm{d}y$$

	$\Pr(X \le x)$					
r	0.01	0.025	0.05	0.95	0.975	0.99
1	0.000	0.001	0.004	3.841	5.024	6.635
2	0.020	0.051	0.103	5.991	7.378	9.210
3	0.115	0.216	0.352	7.815	9.348	11.345
4	0.297	0.484	0.711	9.488	11.143	13.277
5	0.554	0.831	1.145	11.070	12.833	15.086
6	0.872	1.237	1.635	12.592	14.449	16.812
7	1.239	1.690	2.167	14.067	16.013	18.475
8	1.646	2.180	2.733	15.507	17.535	20.090
9	2.088	2.700	3.325	16.919	19.023	21.666
10	2.558	3.247	3.940	18.307	20.483	23.209
11	3.053	3.816	4.575	19.675	21.920	24.725
12	3.571	4.404	5.226	21.026	23.337	26.217
13	4.107	5.009	5.892	22.362	24.736	27.688
14	4.660	5.629	6.571	23.685	26.119	29.141
15	5.229	6.262	7.261	24.996	27.488	30.578
16	5.812	6.908	7.962	26.296	28.845	32.000
17	6.408	7.564	8.672	27.587	30.191	33.409
18	7.015	8.231	9.390	28.869	31.526	34.805
19	7.633	8.907	10.117	30.144	32.852	36.191
20	8.260	9.591	10.851	31.410	34.170	37.566
21	8.897	10.283	11.591	32.671	35.479	38.932
22	9.542	10.982	12.338	33.924	36.781	40.289
23	10.196	11.689	13.091	35.172	38.076	41.638
24	10.856	12.401	13.848	36.415	39.364	42.980
25	11.524	13.120	14.611	37.652	40.646	44.314
26	12.198	13.844	15.379	38.885	41.923	45.642
27	12.879	14.573	16.151	40.113	43.195	46.963
28	13.565	15.308	16.928	41.337	44.461	48.278
29	14.256	16.047	17.708	42.557	45.722	49.588
30	14.953	16.791	18.493	43.773	46.979	50.892