Probability and Statistics

Exercise sheet 2

Exercise 2.1 THE BIRTHDAY PARADOX. Take a urn with N balls numbered with integers from $\{1, ..., N\}$. Perform the experiment of extracting a ball with replacement n times.

(a) Let $A_n :=$ "The first n balls extracted are different". Calculate $\mathbb{P}(A_n)$ (use a Laplace model).

(b) Prove the following inequalities:

$$1 - \frac{n(n-1)}{2N} \leq \mathbb{P}(A_n) \leq \exp\left(-\frac{n(n-1)}{2N}\right).$$

(c) Calculate $n_{\min} = \inf\{n \in \mathbb{N} : \mathbb{P}(A_n) < \frac{1}{2}\}$ for $N = 365$. Relate this problem with the Birthday Problem: " Find the probability that, in a group of n people, there is at least one pair who have the same birthday".

Exercise 2.2 POSTERIOR PROBABILITIES. Suppose that a box contains three coins and that for each coin there is a different probability that we obtain head on a toss. Let p_i denote the probability of a head when the ith coin is tossed ($i = 1, \dots, 3$), and suppose that $p_1 = 1/4$, $p_2 = 1/2$, $p_3 = 3/4$.

(a) Suppose that one coin is selected uniformly at random from the box and when it is tossed once, a head is obtained. What is the posterior probability that the ith coin was selected?

(b) If the same coin were tossed again, what would be the probability of obtaining another head?

(c) Prove the CONDITIONAL BAYES' THEOREM: Let $(A_i)_{i=1\cdots k}$ be a partition of Ω, and B, C are events in Ω,

$$\mathbb{P}(A_i|B \cap C) = \frac{\mathbb{P}(A_i|B)\mathbb{P}(C|A_i \cap B)}{\sum_{j=1}^{k} \mathbb{P}(A_j|B)\mathbb{P}(C|A_j \cap B)}.$$

(d) If the same coin gives another head at the second toss, what is the posterior probability that the ith coin was selected?

(e) Assume that it is always the same coin tossed, and we get always head. What is the recurrence relation of the posterior probability after n tosses that the ith coin was selected?

Exercise 2.3 INTRODUCTION TO BAYESIAN STATISTICS. We have m urns with red and white balls inside. The urn $i \in \{1, ..m\}$ has $2i - 1$ red balls and $2m - 2i + 1$ white ones. We randomly (uniformly) select an urn and extract with replacement n times. Define:

$$X_j := \begin{cases} 1 & \text{If the } j\text{-th ball is red,} \\ 0 & \text{If the } j\text{-th ball is white.} \end{cases}$$

We are interested in the following problem " Given that you see $(X_j)_{j=1}^n$, can you say from which urn the balls where taken?"

(a) Compute $\mathbb{P}(X_1 = x_1, ..., X_n = x_n)$ for $x_i \in \{0, 1\}$. Are $X_1, .., X_n$ independent?

(b) Compute the following probability:

$$\mathbb{P}\left(\text{The urn chosen is } i \mid X_1 = x_1, ..., X_n = x_n\right).$$

Show that this only depends on the number or red balls, i.e., $k = \sum_{i=1}^n x_i$.

(c) Compute $\mathbb{P}\left(\text{The urn chosen is } i \mid X_1 = x_1, ..., X_n = x_n\right)$ for $m = 3$ and $n = 3$.

Exercise 2.4 We are interested in studying the probability of success of a student at an entrance exam to two departments of a university. Consider the following events

$$
\begin{aligned}
A &= \{\text{The student is man}\},\\
A^c &= \{\text{The student is woman}\},\\
B &= \{\text{The student applied for department I}\},\\
B^c &= \{\text{The student applied for department II}\},\\
C &= \{\text{The student was accepted}\},\\
C^c &= \{\text{The student wasn't accepted}\}.
\end{aligned}
$$

We assume that we have the following probabilities (Berkeley 1973):

$\mathbb{P}(A) = 0.73,$
$\mathbb{P}(B \mid A) = 0.69, \quad \mathbb{P}(B \mid A^c) = 0.24,$
$\mathbb{P}(C \mid A \cap B) = 0.62, \quad \mathbb{P}(C \mid A^c \cap B) = 0.82, \quad \mathbb{P}(C \mid A \cap B^c) = 0.06, \quad \mathbb{P}(C \mid A^c \cap B^c) = 0.07.$

(a) Draw a tree describing the situation with the probabilities associated.

(b) Consider the following probabilities $\mathbb{P}[C|A \cap B] = 0.62$, $\mathbb{P}[C|A^c \cap B] = 0.82$, $\mathbb{P}[C|A \cap B^c] = 0.06$, $\mathbb{P}[C|A^c \cap B^c] = 0.07$. With this information, do you think that in this examination women are disadvantaged?

(c) Compute $\mathbb{P}(C \mid A)$ and $\mathbb{P}(C \mid A^c)$. Does this coincide with your answer for b)?

Exercise 2.5 *(Optional)* Let $G = (V, K)$ be an arbitrary finite and undirected graph with vertices V and edges K, i.e., V is a finite set and $K \subseteq \{\{x, y\} \subseteq V : x \neq y\}$. The MAX-CUT problem is to find a subset $A \subseteq V$ such that the number of edges connecting A and A^c (K_A) is maximal, i.e., $K_A = \{\{x, y\} \in K : x \in A, y \in A^c\} \to \max$!. We want to show that there exists $A \subseteq V$ so that $|K_A| \geq \frac{1}{2}|K|$.

(a) Choose $A \subseteq V$ to be a random set uniformly in 2^V (in other words, choose an element A of the power set of V as follows: for every $v \in V$, A contains v with probability 1/2). Calculate $\mathbb{P}(e \in K_A)$, i.e. $\mathbb{P}(\{A : e \in K_A\})$

(b) Using the linearity property of expectation show that

$$\mathbb{E}\left[|K_A|\right] = \frac{1}{2}|K|.$$

(c) Show that there exists an A so that $|K_A| \geq \frac{1}{2}|K|$.

Exercise 2.6 *(Optional)* Monty hall problem. You are on a game show, and you're given the choice of three doors: behind one door is a car; behind the others are goats. You pick a door, say No. 1, and the host, who knows what is behind the doors, opens another, say No. j ($j \in \{2, 3\}$), behind which is a goat. He then tells you, "Do you want to pick door No. l ($l \in \{2, 3\}$, $l \neq j$)?" Is it to your advantage to choose door l ?

To answer this question, we define the following events

$$
\begin{aligned}
B_i &= \text{"The car is behind door i."} && (i = 1, 2, 3),\\
A_j &= \text{"Moderator open door j."} && (j = 2, 3).
\end{aligned}
$$

(a) Define a natural model for the problem. In other words, define Ω and set the probabilities $\mathbb{P}(B_i)$ for $i \in \{1, 2, 3\}$ and $\mathbb{P}(A_j | B_i)$ for $i \in \{1, 2, 3\}$ and $j \in \{2, 3\}$.

(b) Compute, using the Bayes formula, $\mathbb{P}(B_1 \mid A_j)$ for $j \in \{2, 3\}$. Does opening the other door with a goat behind changes the probability that the car is behind door number 1?

(c) Would you modify your choice?