Probability and Statistics

Exercise sheet 3

Exercise 3.1 In a clinical trial with two treatment groups, the probability of success in one treatment group is 0.5, and the probability of success in the other is 0.6. Suppose that there are five patients in each group. Assume that the outcomes of all patients are independent. Calculate the probability that the first group will have at least as many successes as the second group.

Exercise 3.2

(a) Take $p \in [0,1]$ and $n \in \mathbb{N}\backslash\{0\}$. We say that $X \sim Bin(n,p)$ if the distribution of X is

$$\mathbb{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k \in \{0,1,...,n\}.$$

Show that this is indeed a probability distribution using 2 different methods:

1. Calculating $\sum_{k=0}^{n} \mathbb{P}(X=k)$.
2. Representing this probability in terms of the box model with replacement.

Calculate the expected value of X using 2 different methods (the one listed above).

(b) Take $K, n \in \mathbb{N}$ and $N \in \mathbb{N}\backslash\{0\}$ with $K, n \leq N$. We say that a random variable $X \sim Hyp(N,K,n)$ if its distribution is given by

$$\mathbb{P}(X=k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}} \qquad k \in \{\max\{0, n+K-N\}, .., \min\{n,K\}\}$$

Show that this is indeed a probability distribution using 2 different methods:

1. Calculating
$$\sum_{k=\max\{0,n+K-N\}}^{\min\{n,K\}} \mathbb{P}(X=k).$$
Hint: Calculate $(1+x)^n$ in two different ways and identify the terms.
2. Representing this probability in the box model without replacement.

Calculate the expectation using both methods.

Exercise 3.3 A *Poisson process* with rate λ per time unit is a process that satisfies two properties:

1. The number of arrivals in every fixed interval of time of length t has the Poisson distribution with mean λt.
2. The numbers of arrivals in every collection of disjoint time intervals are independent.

Suppose that the arrival time of clients to the store A is a Poisson process with rate 1 per hour.

(a) What is the probability of “the first client comes later than t hours”?

(b) What is the probability of “strictly more than two clients come during the first hour”?

(c) Fix a time $t > 0$, what is the probability of "a client comes at some exact time t"? Does it mean that nobody comes at any time? Is that contradictory?

(d) Suppose that the arrival time of clients to another store B is a Poisson process with rate μ per hour which is independent to the store A, now assumed to have rate λ. What are arrival times of clients to both store (if we forget about in which store they are)?

Exercise 3.4 Let $s \in (1, \infty)$. We define the Riemann's Zeta Function as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

We want to prove that:

$$\zeta(s) = \frac{1}{\prod_{i=1}^{\infty}(1 - p_i^{-s})}.$$

where $p_1, p_2, .., p_k, .. = 2, 3, 5, 7, ...$ is the series of the ordered prime numbers.

(a) Take $(\mathbb{N}, 2^{\mathbb{N}}, \mathbb{P})$ with

$$\mathbb{P}(W) := \frac{1}{\zeta(s)} \sum_{n \in W} \frac{1}{n^s}, \quad W \in 2^{\mathbb{N}}.$$

Show that $(\mathbb{N}, 2^{\mathbb{N}}, \mathbb{P})$ is a probability space.

(b) Let p be a prime number. Define $N_p := \{n \in \mathbb{N} : n \text{ is divisible by p.}\}$. Calculate $\mathbb{P}(N_p)$.

(c) Prove that the events $(N_p)_{p \text{ prime}}$ are independent under this probability measure.

(d) Compute

$$\mathbb{P}\left(\bigcap_{p \text{ prime}} N_p^c\right)$$

and conclude.