Probability and Statistics

Exercise sheet 4

Exercise 4.1 σ-ALGEBRAS.

(a) Let $(\mathcal{A}_i)_{i\in I}$ be a family of σ-algebras. Show that $\bigcap_{i\in I}\mathcal{A}_i$ is a σ-Algebra.

(b) Prove that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are σ-algebras, $\mathcal{A}_1\cup\mathcal{A}_2$ is a σ-Algebra iff $\mathcal{A}_1\subseteq\mathcal{A}_2$ or $\mathcal{A}_2\subseteq\mathcal{A}_3$.

(c) Let $\mathcal{A}$ be a σ-algebra and Ψ an event. For $i\in\mathbb{N}$ define $A_i\in\mathcal{A}$ as " At time i the event Ψ occurs". Write, with the help of the A_i's the following sets. Additionally show that they belong to $\mathcal{A}$.

1. " Ψ never occurs"
2. " Ψ occurs infinitely many times".
3. " From a point in time onward Ψ never occurs".
4. " Ψ occurs exactly twice".
5. " Ψ occurs in total an odd number of times".

Which of them belong to the tail σ-algebra, i.e.,

$$\mathcal{A}_\infty := \bigcap_{n\in\mathbb{N}} \sigma(\{A_k : k\geq n\})?$$

Exercise 4.2 BOREL CANTELLI.

(a) Construct a probability space $(\Omega,\mathcal{A},\mathbb{P})$ and a series of measurable sets $(A_n)_{n\in\mathbb{N}}$ with $\sum_{n\in\mathbb{N}}\mathbb{P}(A_n)=\infty$ and $\mathbb{P}\left(\bigcap_{n\in\mathbb{N}}\bigcup_{k\geq n}A_k\right)=0$.

(b) Let $(\Omega,\mathcal{A},\mathbb{P})$ be a probability space. Take $(U_n)_{n\in\mathbb{N}}$ a series of uniform independent random variables on $(0,1)$, i.e., for $0\leq x\leq 1$, $\mathbb{P}(U_n\in[0,x])=x$.

(i) Show that:

$$\mathbb{P}\left((\exists\alpha>1)\liminf n^\alpha U_n\in\mathbb{R}\right)=0.$$

Hint: It may be useful to define, for $\alpha>1$ $A_n^\alpha := \{U_n < n^{-\alpha}\}$. Do not forget that the countable union of sets of probability 0 has probability 0.

(ii) Prove that:

$$\mathbb{P}\left(\liminf nU_n\in\mathbb{R}\right)>0.$$

Exercise 4.3 Let $(\{0,2\}^{\mathbb{N}},\mathcal{A},\mathbb{P})$ be the model of infinite tossing of coins (Lecture notes Satz 3.2, p. 37). We consider the random variable:

$$\begin{array}{llcl} X: & \{0,2\}^{\mathbb{N}} & \longrightarrow & [0,1] \\ & \omega=(\omega_1,\omega_2,\ldots) & \mapsto & X(\omega)=\sum_{n=1}^{\infty}\frac{\omega_n}{3^n}. \end{array}$$

(a) Prove that X is measurable.

(b) Show that the cumulative distribution function of X is continuous.

(c) Prove that there exist disjoint intervals $I_k \subseteq [0,1]$ so that F is constant in I_k and $\lambda(\bigcup_{k=1}^{\infty} I_k) = 1$. (Where λ is the Lebesgue measure)
Hint:

- $X(\omega) = \sum_{n=1}^{\infty} \frac{X_n(\omega)}{3^n}$.
- F is constant on $X(\{0,2\}^n)^c$.