Probability and Statistics

Exercise sheet 5

Exercise 5.1 THE BERTRAND'S PARADOX. Consider an equilateral triangle inscribed in a circle of radius 1. Suppose a chord of the circle is chosen at random. What is the probability that the chord is longer than a side of the triangle?. To solve this, try the following probability models:

(a) The "random endpoints" method: Choose two uniform random points on the circumference of the circle and draw the chord joining them, i.e., let $U, V \sim U(0,1)$, define $X = e^{U2\pi i}, Y = e^{V2\pi i}$ and take the chord connecting X and Y.

(b) The "random radius" method: Choose a radius of the circle, choose a uniform point on the radius and construct the chord through this point and perpendicular to the radius, i.e., choose a radius and choose $r \sim U(0,1)$, take the point on the radius that is at distance r from the center and the chord will be the only chord perpendicular to this point.

(c) The "random midpoint" method: Choose a point uniformly anywhere within the circle and construct a chord with the chosen point as its midpoint, i.e., take $(x,y) \sim U(B(0,1))$ and take the chord whose midpoint is (x,y).

(d) Is this a contradiction?

Exercise 5.2 Let X_1 and X_2 be two random variables following a normal distribution with mean μ_1 and variance σ_1^2 (resp. with mean μ_2 and variance σ_2^2). Prove that if X_1 is independent of X_2 then $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Exercise 5.3 Let X be a standard normal random variable.

(a) Prove that if we take $Y := X^2$, then $f_Y(y) = ce^{-y/2}y^{-1/2}\mathbb{1}_{y\geq 0}$ (We say that Y is distributed according to a χ-squared with one degree of freedom).

(b) If Y_1 and Y_2 are two independent copies of Y, prove that $f_{Y_1+Y_2}(x) = c_2e^{-x/2}\mathbb{1}_{x\geq 0}$. What is the name of this distribution.

(c) With the help of induction prove that $\sum_{i=1}^n Y_i$, where $(Y_i)_{i=1}^n$ are independent copies of Y, has as a density function

$$f_{\sum_{i=1}^n Y_i}(x) = c_n x^{\frac{n}{2}-1}e^{-\frac{x}{2}}\mathbb{1}_{\{x\geq 0\}}.$$

This is call a χ-squared distribution with n degrees of freedom.

Exercise 5.4 MEMORYLESSNESS OF EXPONENTIAL RANDOM VARIABLES. We say that a random variable X has an exponential distribution of parameter λ (write it $\mathcal{E}(\lambda)$) if for all $t \geq 0$,

$$\mathbb{P}(X \geq t) = e^{-\lambda t}.$$

(a) Find the density function (with respect to the Lebesgue Measure) of an exponential random variable. Calculate its mean and its variance.

(b) Show that if $X_1 \sim \mathcal{E}(\lambda_1)$, $X_2 \sim \mathcal{E}(\lambda_2)$ and X_1 is independent of X_2, then $\min\{X_1, X_2\} \sim \mathcal{E}(\lambda_1 + \lambda_2)$.

(c) Show that

$$\mathbb{P}\left(X \geq t+h \mid X \geq h\right) = \mathbb{P}(X \geq t).$$

This is called the memoryless property of exponential random variables. We want to prove that the only random variable that has this property is the exponential random variable. Suppose that $Y : \Omega \mapsto \mathbb{R}^+$ has the memoryless property, i.e.,

$$\mathbb{P}\left(Y \geq t+h \mid Y \geq h\right) = \mathbb{P}(Y \geq t).$$

(d) Define $G(t) := \mathbb{P}(Y \geq t)$ and prove that $G(t+h) = G(t)G(h)$.

(e) Prove that for all $m, n \in \mathbb{N}$, $G\left(\frac{m}{n}\right) = G(1)^{\frac{m}{n}}$.

(f) Using the monotone property of G prove that for all $t \geq 0$, $G(t) = G(1)^t$. Conclude that Y has an exponential distribution and give its parameter.

Exercise 5.5 *(optional: conditional expectation and tower property)* Suppose that internet users access a particular Web site according to a Poisson process with rate Λ per hour, but Λ is unknown. The Web site maintainer believes that Λ has a continuous distribution with probability density function:

$$f(\lambda) = \begin{cases} 2e^{-2\lambda} & \text{for } \lambda > 0 \\ 0 & \text{otherwise} \end{cases}.$$

Let X be the number of users who access the Web site during a one-hour period. If $X = 1$ is observed, find the conditional probability density function of Λ given $X = 1$.