Probability and Statistics

Exercise sheet 6

Exercise 6.1 Take X_n i.i.d random variable so that

$$\mathbb{E}(X_1) = 1, \quad \mathrm{Var}(X_1) = 2,$$

and define $S_n := \sum_{i=1}^n X_i$.

(a) Use Chebyshev-inequality to estimate

$$\mathbb{P}\left(\left|\frac{S_n}{n} - 1\right| \leq 0.5\right).$$

What is the value of the bound when $n = 40$.

(b) Use the Central Limit Theorem to estimate

$$\mathbb{P}\left(\left|\frac{S_n}{n} - 1\right| \leq 0.5\right).$$

What is the value of the bound when $n = 40$.

Exercise 6.2 Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $(Z_n)_{n\in\mathbb{N}}$ a sequence of random variables.

(a) Prove that if $Z_n \xrightarrow{\mathbb{P}} c \in \mathbb{R}$, then for all bounded and continuous functions f

$$\mathbb{E}(f(Z_n)) \to f(c).$$

(b) Show that if $Z_n \to c \in \mathbb{R}$ in distribution, then $Z_n \xrightarrow{\mathbb{P}} c$.

Exercise 6.3 Consider the probability space $(\Omega, \mathcal{A}, \mathbb{P}) = ([0,1], \mathcal{B}([0,1]), \lambda|_{[0,1]})$, where $\lambda|_{[0,1]}$ is the Lebesgue measure over $[0,1]$. Let $X_n(\omega) = \mathbb{1}_{A_n}(\omega)$ be a sequence of random variables with $A_n \in \mathcal{B}([0,1])$.

(a) What condition on the sets $(A_n)_{n\in\mathbb{N}}$ is necessary for the convergence in probability under $\mathbb{P}$ of the sequence $(X_n)_{n\in\mathbb{N}}$ to 0 ?

(b) Write the event $\{\omega : X_n(\omega) \to 0\}$ with the sets $(A_n)_{n\in\mathbb{N}}$.

(c) Find a sequence $(A_n)_{n\in\mathbb{N}}$ of events such that $X_n \xrightarrow{\mathbb{P}} 0$ but $\{\omega : X_n(\omega) \to 0\} = \emptyset$.

Exercise 6.4 Let $(X_i)_{i\geq 1}$ be a sequence of random variables with

$$\begin{aligned}
&\mathbb{E}(X_i) = \mu \quad \forall i,\\
&\mathrm{Var}(X_i) = \sigma^2 < \infty \quad \forall i,\\
&\mathrm{Cov}(X_i, X_j) = R(|i-j|) \quad \forall i, j,
\end{aligned}$$

where R is a real function. Define $S_n := \sum_{i=1}^n X_i$.

(a) Prove that if $\lim_{k\to\infty} R(k) = 0$ then $\lim_{n\to\infty} \frac{S_n}{n} = \mu$ in probability.

(b) Prove that if $\sum_{k\in\mathbb{N}} |R(k)| < \infty$ then $\lim_{n\to\infty} n\mathrm{Var}(\frac{S_n}{n})$ exists.

Exercise 6.5 *(Optional)*

(a) Let μ_n and ν_n be two sequences of probability measure on $\mathbb{R}$, and $\epsilon_n \in (0,1)$ with $\epsilon_n \to 0$. Prove that if $\mu_n \to \mu$ in distribution, then $(1-\epsilon_n)\mu_n + \epsilon_n\nu_n \to \mu$ in distribution.

(b) Construct with the help of (a) a sequence μ_n so that $\mu_n \to \mu$ in distribution but $\lim_{n\to\infty} \int |x| d\mu_n(x) \neq \int |x| d\mu(x)$.

(c) Prove that if $\mu_n \to \mu$ in distribution and $\sup_n \int x^2 d\mu_n(x) = K < \infty$ then

$$\int |x| d\mu_n(x) \to \int |x| d\mu(x).$$

Hint: For all M prove that

$$\int \min\{|x|, M\} d\mu_n(x) \to \int \min\{|x|, M\} d\mu(x).$$

and that

$$0 \leq \int |x| d\mu_n(x) - \int \min\{|x|, M\} d\mu_n(x) \leq K/M.$$