Probability and Statistics

Exercise sheet 7

Exercise 7.1

(a) Let $X \sim \mathcal{E}(\lambda)$ be an exponential random variable with parameter λ. Compute the α-quantile for all $\alpha \in (0,1)$.

Note: The α-quantile q_α is a real number such that

$$\mathbb{P}(X \leq q_\alpha) \geq \alpha \text{ and } \mathbb{P}(X \geq q_\alpha) \geq 1-\alpha.$$

(b) Let U be a uniformly distributed random variable in $\{1, 2, \cdots, N\}$. Determine the CDF of U, compute the α-quantile for all $\alpha \in (0,1)$. For which α is the quantile uniquely determined?

Exercise 7.2 Let X and Y be random variables with joint density distribution given by

$$f_{X,Y}(x,y) = e^{-x^2 y} \mathbb{1}_{\{x \geqslant 1\}} \mathbb{1}_{\{y \geqslant 0\}}$$

(a) Why is this a probability measure?

(b) What is the density function of X.

(c) Compute $\mathbb{P}(Y \leq 1/X^2)$.

Exercise 7.3 Let X and Y be two independent exponential random variables with parameter λ. Let $a > 0$.

(a) What is the joint density of the couple of random variables $(X, X+Y)$?

(b) Let $b \leq a$, what is the probability of $X \leq b$ conditioned on the event

$$B := \{X \leq a \text{ and } X + Y \geq a\}.$$

(c) What is the conditional density of X given the event B?

Exercise 7.4

(a) Take X a random variable. Prove that for all $\lambda \geq 0$

$$\mathbb{P}(X \geq t) \leq \exp(-\lambda t)\mathbb{E}\left(\exp(\lambda X)\right).$$

(b) Define $\phi_X(\lambda) := \ln\left(\mathbb{E}\left(e^{\lambda X}\right)\right)$. Prove that $\phi(\lambda) \geq \lambda \mathbb{E}(X)$.

(c) Prove that

$$\mathbb{P}(X \geq t) \leq \exp(-\sup_{\lambda \geq 0}\{\lambda t - \phi_X(\lambda)\}).$$

(d) If $X \sim N(0, \sigma)$, calculate $\phi_X(\lambda)$.

(e) Prove that if X is a positive random variable

$$\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X \geq t)dt$$

(f) Show that if $X \sim N(0, \sigma)$ and Y is a random variable such that $\phi_Y(\lambda) \leq \phi_X(\lambda)$, then

$$\mathbb{E}(Y^2) \leq 4\sigma^2.$$

Exercise 7.5 Let X be a real-valued random variable. We define the characteristic function of X by

$$\begin{aligned} \varphi_X &: \mathbb{R} \to \mathbb{C} \\ t &\mapsto \varphi_X(t) := \mathbb{E}[e^{itX}] = \int e^{itx} \mu(dx), \end{aligned}$$

in which μ is the distribution of X on $\mathbb{R}$. It represents an important analytic tool, that the distribution of a random variable is uniquely determined (characterized) by the characteristic function. Show the following features:

(a)
- $\varphi_X(0) = 1$,
- $|\varphi_X(t)| \leq 1$,
- φ_X is continuous, and
- $\varphi_{aX+b}(t) = e^{itb}\varphi_X(at)$ for all $a, b \in \mathbb{R}$.

(b) Show that if the n-th moment of X exists, i.e. $\mathbb{E}[|X|^n] < \infty$, then φ_X is n times differentiable, and

$$\varphi_X^{(k)}(t) = i^k \mathbb{E}[X^k e^{itX}], \quad \text{for all } k \leq n,$$

(in particular $\varphi_X^{(k)}(0) = i^k \mathbb{E}[X^k]$). *Hint:* one can use the inequality $|\frac{e^{i\alpha}-1}{\alpha}| \leq 1$, ($\alpha \in \mathbb{R}$).

(c) Compute the characteristic function for the standard normal distribution $\mathcal{N}(0,1)$, then for $\mathcal{N}(\mu, \sigma^2)$.

(d) Let X and Y be two independent random variables, defined on the same probability space. What is the characteristic function of $X + Y$?

Exercise 7.6 Strong law of large number for variable with 4th moment. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space . Take $(X_n)_{n\in\mathbb{N}}$ a series of independent identically distributed random variables. Suppose that $\mathbb{E}(X_1) = 0$ and $\mathbb{E}\left(X_1^4\right) < \infty$, and define $S_n = \frac{1}{n}\sum_{i=1}^n X_i$.

(a) Prove that $\mathbb{E}(S_n^4) = \frac{1}{n^3}\mathbb{E}\left(X_1^4\right) + \frac{6(n-1)}{n^3}\mathbb{E}\left(X_1^2\right)^2$. Why does $\mathbb{E}\left(X_1^2\right) < \infty$ hold ?

(b) Show that

$$\mathbb{P}\left(|S_n| > a\right) \leq \frac{6}{a^4}\frac{1}{n^2}\mathbb{E}\left(X^4\right).$$

(c) Using Borel-Cantelli show that $\mathbb{P}\left(\lim_{n\to\infty} S_n = 0\right) = 1$.

(d) Now if the hypothesis $\mathbb{E}(X_1) = 0$ is changed. Prove that $\lim_{n\to\infty} S_n = \mathbb{E}(X_1)$.