Probability and Statistics

Exercise sheet 8

Exercise 8.1 Let X and Y two independent standard normal random variables ($\mathcal{N}(0,1)$). Define the random variable

$$Z := \begin{cases} X & \text{if } Y \geq 0, \\ -X & \text{if } Y < 0. \end{cases}$$

(a) Compute the distribution of Z.

(b) Compute the correlation between X and Z.

(c) Compute $\mathbb{P}(X + Z = 0)$.

(d) Does (X, Z) follow a multivariate normal distribution (in other words, is (X, Z) a Gaussian vector)?

Exercise 8.2 Assume that $X := (X_1, X_2, \cdots, X_n)$ is a Gaussian vector with mean 0 and covariance matrix K_X, defined by

$$K_X(i,j) = Cov(X_i, X_j).$$

(a) Let $\alpha_1, \cdots, \alpha_n$ be n real numbers, what is the law of $\sum_{i=1}^n \alpha_i X_i$ in terms of K_X?

(b) What can you say about K_X?

(c) If $K_X(1,2) = 0$, show that X_1 and X_2 are independent. Is that true if we don't assume that (X_1, X_2) is a Gaussian vector?

Hint: The characteristic function of the pair of random variables $X := (X_1, X_2)$, which is defined as

$$\begin{aligned} \varphi_X &: \mathbb{R}^2 \to \mathbb{C} \\ t = (a,b) &\mapsto \varphi_X(t) := \mathbb{E}[\mathrm{e}^{it \cdot X}] = \mathbb{E}[\mathrm{e}^{i(aX_1 + bX_2)}], \end{aligned}$$

characterizes also the joint law of (X_1, X_2).

Exercise 8.3 Suppose two random variables X_1 and X_2 have a continuous joint distribution for which the joint probability density function (p.d.f.) is given by

$$f(x_1, x_2) = \begin{cases} 4x_1x_2 & \text{for } 0 < x_1, x_2 < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $\mathrm{Cov}(X_1, X_2)$. Determine the joint p.d.f. of two new random variables Y_1 and Y_2, which are defined by the relations

$$Y_1 = \frac{X_1}{X_2} \text{ and } Y_2 = X_1 X_2.$$

Exercise 8.4 Let α and β be positive numbers. A random variable X has the *gamma distribution with parameters α and β* if X has a continuous distribution for which the probability density function is

$$f(x|\alpha,\beta) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

Γ is the function defined as: for $\alpha > 0$,

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

Consider X a gamma-distributed random variable with parameters α and β.

(a) For $k = 1, 2, \ldots$, show that the k-th moment of X is

$$\mathbb{E}(X^k) = \frac{\Gamma(\alpha+k)}{\beta^k \Gamma(\alpha)} = \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\beta^k}.$$

What are $\mathbb{E}(X)$ and $\mathrm{Var}(X)$?

(b) What is the moment generating function of X?

(c) If the random variables $X_1, \cdots, X_k$ are independent, and if X_i (for $i = 1, \ldots, k$) has the gamma distribution with parameters α_i and β, show that the sum $X_1 + \cdots + X_k$ has the gamma distribution with parameters $\alpha_1 + \cdots + \alpha_k$ and β.

Exercise 8.5 Service Times in a Queue. For $i = 1, \cdots, n$, suppose that customer i in a queue must wait time X_i for service once reaching the head of the queue. Let Z be the rate at which the average customer is served. A typical probability model for this situation is to say that, conditional on $Z = z$, $X_1, \ldots, X_n$ are i.i.d. with a distribution having the conditional p.d.f. $g_1(x_i|z) = z\exp(-zx_i)$ for $x_i > 0$. Suppose that Z is also unknown and has the p.d.f. $f_2(z) = 2\exp(-2z)$ for $z > 0$.

(a) What is the joint p.d.f. of $X_1, \ldots, X_n, Z$.

(b) What is the marginal joint distribution of $X_1, \ldots, X_n$.

(c) What is the conditional p.d.f. $g_2(z|x_1, \ldots, x_n)$ of Z given $X_1 = x_i, \ldots, X_n = x_n$? For this we can set $y = 2 + \sum_{i=1}^n x_i$.

(d) What is the expected average service rate given the observations $X_1 = x_1, \cdots, X_n = x_n$?

Exercise 8.6

(a) Compute the limit of $\lim_{n\to\infty} e^{-n} \sum_{k=0}^n \frac{n^k}{k!}$
Hint: You can use the central limit theorem for $(X_i)_{i\in\mathbb{N}}$ i.d.d. random variables such that $X_i \sim Poi(1)$.