Probability and Statistics

Exercise sheet 9

Exercise 9.1 If $\log(X)$ has the normal distribution with mean μ and variance σ^2, we say that X has the *lognormal distribution* with parameters μ and σ^2.

A popular model for the change in the price of a stock over a period of time of length u is to say that the price after time u is $S_u = S_0 Z_u$, where Z_u has the lognormal distribution with parameter μu and $\sigma^2 u$. In this formula, S_0 is the present price of the stock, and σ is called the *volatility* of the stock price.

(a) What is the expected value of S_1?

(b) Find the distribution of $1/S_1$.

(c) What is the expected value of $1/S_1$?

(d) What are k-th moments of S_1, for $k = 1, 2, \ldots$?

Exercise 9.2 Suppose that Z has the standard normal distribution, V has the χ-squared distribution with n degrees of freedom, and that Z and V are independent. Let

$$T = \frac{Z}{\sqrt{V/n}}.$$

You will show that T has the probability densuty function given by

$$f(t) = \frac{\Gamma((n+1)/2)}{\sqrt{n\pi}\Gamma(n/2)} \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2} \qquad t \in \mathbb{R}.$$

Recall that we have seen in Series 4, that the probability density function of a χ-squared with n degrees of freedom is, for some $c_n \in \mathbb{R}$:

$$f_V(x) = c_n x^{n/2-1} e^{-x/2} \mathbb{1}_{\{x \geq 0\}}.$$

(a) Find the joint probability density function of (T, V).

(b) Show first that the conditional distribution of T given $V = v$ is normal with mean 0 and variance $\frac{n}{v}$.

(c) Compute c_n.

(d) Find the probability density function of T.

Exercise 9.3 We would like to compute

$$A := \int_{-3}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

using Monte-Carlo method:

(a) Express A under the form $\mathbb{E}[f(U)]$, where U is a standard Gaussian random variable, and f an appropriate function.

(b) Take $(U_i)_{i\in\mathbb{N}}$ an i.i.d. family having the same law as U. Set

$$S_n := \frac{1}{n}\sum_{i=1}^{n} f(U_i).$$

What is the distribution of $S_n - A$?

(c) Compute $\mathbb{E}(S_n)$ and show that $\mathrm{Var}(S_n) = (A - A^2)/n$.

(d) Show that for any $x > 0$, $\mathbb{P}[|S_n - A| \geq x] \leq 1/nx^2$, thus converges to 0 when $n \to \infty$.

(e) Which theorem can you apply to get directly the above convergence?

Exercise 9.4 If a random variable X has the χ^2 distribution with m degrees of freedom, then the distribution of $X^{1/2}$ is called a *chi (χ) distribution with m degrees of freedom*. Determine the mean of this distribution.