Hints and Answers to Selected Exercises in Fundamentals of Probability: A First Course, Anirban DasGupta, Springer, 2010

Chapter 1

1.1. The number of sample points are: a) 32760; b) 1560; c) 6840; d) 9; e) ∞.

1.3. 0.7748.

1.4. 10/21.

1.5. Yes, because the total possible number of initials is less than 20,000.

1.6. a) $P(AB) + P(BC) + P(AC) - 3P(ABC)$.

1.7. a) EF^cG^c; b) EGF^c; d) $E \cup F \cup G$.

1.9. a) Mutually exclusive; c) Not mutually exclusive.

1.10. a) Head on the first toss; c) Same outcome on the first two tosses.

1.11. 3/4.

1.13. 1/12.

1.14. 5/6.

1.15. $(3!)^4/9!$.

1.16. a) 10^{20}; b) $20!/2^{10}$.

1.18. 35/216.

1.21. 1/3.

1.22. Use inclusion-exclusion formula with A_i as the event that the handyman gets day i of the week off, $i = 1, 2, \cdots, 7$.

1.23. Just count the sample points.

1.25. Use inclusion-exclusion formula with A_i as the event that the ith child gets chosen everyday of the week. For example, $P(A_1) = (2/3)^7$.

1.26. 1/2.

1.30. $\frac{(8!)^2}{64\times 63\times 62\times\cdots\times 57}$.

1.31. 3/8; 1/2; 19/32; 43/64.

1.33. (a) is more likely.

1.35. $1 - \frac{\binom{7}{6}}{\binom{10}{6}}$.

1.36. $\frac{6!12!(3^6)}{18!}$.

1.37. Think of the complement of the event, which is that the $2m$ shoes are of $2m$ different colors, and from each of these $2m$ colors, you would choose either the left or the right shoe.

1.40. $\frac{(n-r+1)r!(n-r)!}{n!}$.

1.41. $\frac{2(n-k+1)}{(n+1)(n+2)}$.

1.43. a) $\frac{\binom{26}{13}}{\binom{52}{13}}$; b) $\frac{\binom{26}{13}}{\binom{52}{13}\binom{39}{13}}$.
1.44. Use inclusion-exclusion formula with A_i as the event that the hand of the ith player is void in the particular suit. For example, $P(A_1) = \frac{\binom{39}{13}}{\binom{52}{13}}$.
1.47. 1/4.

Chapter 2

2.1. $n \geq 5$.
2.2. Solve the equation $\frac{m(m-1)\cdots(m-n+1)}{m^n} = 0.5$ with $m = (365)^3$. Use Stirling's approximation to find the root.

Chapter 3

3.1. 5/6.
3.2. 5/6.
3.3. 1/3.
3.4. $P(B|A) \geq 8/9$.
3.6. 2.25% of days.
3.8. 10/19.
3.9. Consider the events A_i = North has i aces and South has $4-i$, and look at $P(A_1|\cup_{i=1}^4 A_i)$ and $P(A_3|\cup_{i=1}^4 A_i)$.
3.10. 0.0006.
3.11. $\frac{\binom{39}{2}}{\binom{59}{2}}$.
3.12. The winning probability of Sam is $.2 \times \sum_{j=0}^{\infty}(0.8 \times 0.7 \times 0.6)^j$. Find one of the other two probabilities, and then the third one by subtraction.
3.15. Consider the events A_i = The last face to show up is face i, and observe that $P(\cup_{i=1}^6 A_i) = 6P(A_1)$.
3.16. First solve the equation $p^4 + (1-p)^4 = 0.25$.
3.18. 5/7.
3.19. 51293 weeks.
3.20. Both strategies are equally good.
3.21. 1/18; use Bayes' theorem.
3.22. $x/(x+y)$.
3.24. $p/(6-5p)$.
3.26. a) 0.0011; b) 0.0441.

3.27. 0.2308.
3.28. Consider the probability that the first cell remains empty and the other two do not, and prove that this probability is $3y(1-x)(1-x-y)$. Then find the required conditional probability.
3.29. List the sample points and find $P(b^2 < ac)$, where b is the off-diagonal element, by direct counting.
3.30. 0.9375.
3.33. The probability of this event is about 0.0182, which is small.
3.34. $\frac{(N-r)(N-r-1)}{(N-1)(N-2)}$.

Chapter 4

4.1. $p(0) = p(4) = 1/16; p(1) = p(3) = 1/4; p(2) = 3/8.$
$F(x) = 0(x < 0); = 1/16(0 \leq x < 1); = 5/16(1 \leq x < 2); = 11/16(2 \leq x < 3); = 15/16(3 \leq x < 4); = 1(x \geq 4).$
4.2. $p(0) = p(3) = 1/6; p(1) = 5/18; p(2) = 2/9; p(4) = 1/9; p(5) = 1/18.$
4.3. $p(0) = 1/2; p(1) = 1/4; p(2) = 1/8; p(3) = p(4) = 1/16.$
4.4. Use the total probability formula; e.g., $P(X = 1) = \frac{1}{6}\sum_{n=1}^{6} \frac{n}{2^n}$, etc.; $E(X) = 7/4$.
4.5. 6.5
4.7. (a) $P(h(X) = \pm 1) = 4/13; P(h(X) = 0) = 5/13; E(h(X)) = 0.$
4.8. First prove that the variance of X^2 must be zero.
4.10. The expected value is $\frac{3n(n-1)5^{n-2}}{6^n}$.
4.12. (a) 2/5; (b) 4/3.
4.14. 6.
4.15. $\frac{n-2}{216}$.
4.17. Consider indicator random variables I_i of the events that the pair of cards marked as i remain in the jar. Show that $E(I_i) = \frac{\binom{2N-2}{m}}{\binom{2N}{m}}$.
4.20. Show that $P(X \leq x) = 0$ for $x < 5$ and is equal to $\frac{\binom{x}{5}}{\binom{10}{5}}$ for $x = 5, 6, \cdots, 10$. From here, find the pmf of X.
4.21. Show that $P(X > n) = \frac{365 \times 364 \times \cdots \times (366-n)}{(365)^n}$ for $n = 2, 3, \cdots, 365$, and then apply the tailsum formula.
4.22. 1.94×10^{-5}.
4.23. $\log 2$.

4.26. 5.5.
4.27. Differentiate $E(X-a)^2$ with respect to a.
4.28. Write $\sum_{n=1}^{\infty} nP(X>n) = \sum_{n=1}^{\infty} n(\sum_{j=n+1}^{\infty} p(j))$ as $\sum_{j=2}^{\infty} p(j)(\sum_{n=1}^{j-1} n)$, and simplify.
4.29. Mean $=(n+1)/2$; variance $=(n^2-1)/12$. Find medians separately for odd and even n.
4.31. (a) At most 111,111 people; (b) at most 55,555 people.
4.33. First show that the maximum possible value of $E(X^2)$ if $E(X)=\mu$ is μM. Then apply the Paley-Zygmund inequality.
4.34. Look at random variables whose pmf go to zero at the rate $\frac{1}{x^{p+1}}$.
4.37. Show that $\{f(X_1)\leq c\}$ and $\{g(X_2)\leq d\}$ are independent events for any c, d.
4.38. A sufficient condition is that $E(X_1)=E(X_2)=0$.

Chapter 5

5.1. pgf is $s/(2-s), |s|<2$; the mgf is $e^t/(2-e^t), t<\log 2$.
5.2. For example, $G(s)=\sqrt{s}, s\geq 0$.
5.4. $e^{bt}\psi(at)$.
5.5. Use the fact that $\sqrt{x}\leq x$ for all $x\geq 1$.
5.7. Use the fact that $E[X(X-1)\cdots(X-k+1)]=G^{(k)}(1)$. For example, $\sigma^2=G^{(2)}(1)+G^{(1)}(1)-[G^{(1)}(1)]^2$.
5.9. Use the fact that e^{tx} is a convex function of x for any t.
5.11. $\kappa_1=p; \kappa_2=p(1-p); \kappa_3=p-3p^2+2p^3; \kappa_4=p-7p^2+12p^3-6p^4$.

Chapter 6

6.1. $0.2461(n=10); 0.1445(n=30); 0.1123(n=50)$. The limit can be proved to be zero.
6.2. $p(0)=p(3)=0.12; p(1)=0.38$. The distribution is not Binomial.
6.3. 0.5367; 0.75. The most likely value is zero.
6.5. $0.6651(n=1); 0.6187(n=2); 0.5973(n=3); .5845(n=4); 0.5757(n=5)$.
6.8. For $p\geq .5$.
6.9. 15/16.
6.10. 0.0107.

6.11. $(k/N)^n - ((k-1)/N)^n, k = 1, 2, \cdots, N$ for with replacement sampling; $\frac{\binom{k}{n} - \binom{k-1}{n}}{\binom{N}{n}}, k = n, \cdots, N$ for without replacement sampling.
6.12. Compute a Poisson approximation with $n = 100$ and $p = \frac{\binom{200}{100}}{2^{200}} = 0.05635$. The final answer is 0.0611.
6.14. The probability of getting an A is $(0.2)^5$; the probability of getting a B is $\binom{5}{4}(0.2)^5(0.8) + \binom{6}{4}(0.2)^5(0.8)^2$, etc.
6.16. Binomial(200, .75).
6.19. Find the sum $\sum_{k=0}^{\infty} P(X = Y = k)$ and simplify it to $e^{-\lambda}(1 - p)^n n!\left(\sum_{k=0}^{n} (\frac{\lambda p}{1-p})^k / [(k!)^2 (n-k)!] \right)$.
6.20. Solve two appropriate equations to first show that $p = 0.9881, n = 86$, and so the variance is $np(1-p) = 1.01$.
6.23. 0.0707.
6.24. 0.1538.
6.26. 0.0248.
6.27. Use the skewness and kurtosis formulas for a general Poisson distribution; the answers are 0.4472; 0.5.
6.28. No. You have to first prove that if three such values exist, they must be consecutive integers.
6.30. Compute a Poisson approximation to $P(X \geq 2)$. The answer is 1.7993×10^{-7}.
6.32. 0.0479.
6.33. No.
6.34. $\frac{1+e^{-2\lambda}}{2}$.
6.37. (a) 0.1839; (b) 0.0803; (c) 0.9810.
6.38. 0.2212.
6.42. Expand $(q+p)^n$ and $(q-p)^n$ by using the Binomial expansion theorem and add, with $q = 1 - p$.

Chapter 7

7.1. It is a density for $k = 2$.
7.2. $c = 1$.
7.3. (a) $c = 2$; (b) For $x > 0, F(x) = 1/2 + x^2 - x^4/2$; (c) 0.28125; 0.28125; 0.4375.

7.4. $F(x) = 0(x < 0), = p(1-\cos(x)) + (1-p)\sin x(0 \leq x \leq \frac{\pi}{2}), = 1(x > \frac{\pi}{2})$. Median solves $p(1-\cos(x)) + (1-p)\sin x = 0.5$. If $p = 0.5$, the median is $\frac{\pi}{4}$.
7.6. (a) $c = 4/16875$; (b) 40.74%; 11.11%; (c) 9 minutes.
7.7. $F(x) = 0(x < 0), = x/5(0 \leq x \leq 2.5), = 1(x > 2.5)$; expected payout is \$1875.00. This distribution is a mixed distribution.
7.9. Density function of the ratio of the two lengths is $\frac{2}{1+v^2}, 0 < v < 1$; expected value is $\log 2 = 0.6931$. You should find the density by first finding the CDF.
7.10. One example is $f(x) = \frac{1}{2\sqrt{x}}, 0 < x < 1$.
7.11. One example is $f(x) = \frac{1}{x^2}, x \geq 1$.
7.13. Quantile function is $\frac{1}{1-p}$.
7.16. (a) Mean $= 0$; variance $= 1/2$.
7.17. $c = 12/5$; mean is $\frac{13\pi}{100}$ and the variance is $\frac{5\pi^2}{224} - \frac{169\pi^2}{10000}$.
7.18. Quantile function is $\tan\left[(p - \frac{1}{2})\pi\right]$. 75th percentile equals 1.
7.20. Use the trigonometric identity $\arctan x + \arctan y = \pi + \arctan\frac{x+y}{1-xy}$ for all $x, y > 0, xy > 1$, and plug in suitable special values of x, y.
7.21. The density function of $Y = \frac{1}{X^2}$ is $\frac{e^{-\frac{1}{2y}} y^{-3/2}}{\sqrt{2\pi}}, y > 0$.
7.23. First prove that $E[f(X)]$ must belong to the range space of f and then use the intermediate value theorem. In the standard normal case, $x_0 = \sqrt{\log 2}$.
7.24. 4.761905.
7.25. Use the formula for the area of a triangle in pp 439 of the text.
7.27. $\frac{2^{\alpha/2}\Gamma(\frac{\alpha+1}{2})}{\sqrt{\pi}}$.
7.30. $\mu = \int_0^\infty [e^{-\int_0^x h(t)dt}]dx$.
7.32. $\frac{n}{n+1}$.
7.33. $\frac{1}{n}$.
7.35. Use Lyapounov's inequality suitably.
7.36. Use Jensen's inequality suitably.
7.37. $I(x) = 0(x \leq 1), = x - \log x - 1(x > 1)$.

Chapter 8

8.1. $a = 1/2, b = 1/4$.
8.2. $a = -1, b = 5$.

8.3. 1/3.

8.5. (a) $x^3 - 3x$ is a strictly monotone function of x on $(0, 1)$, taking values in $(-2, 0)$. The equation $x^3 - 3x = y$ has a unique real root, say $x = g(y)$. The density function of $Y = X^3 - 3X$ is $|g'(y)|, -2 < y < 0$.

(b) $f_Y(y) = \frac{1}{\sqrt{y}}, 0 < y < 1/4$;

(c) $f_Y(y) = \frac{1}{2\pi y^{3/4}\sqrt{1-\sqrt{y}}}, 0 < y < 1$.

8.6. (a) X can be simulated as $X = U^{1/5}$;

(b) X can be simulated as $X = \left(\sin \frac{\pi}{2} U\right)^2$;

(c) X can be simulated as $X = 2 - \log\left(2(1-U)\right)$ if $U > \frac{1}{2}$ and $X = 2 + \log\left(2U\right)$ if $U \leq \frac{1}{2}$.

8.7. Substitute $\beta = 26 - \alpha$ into the equation $F(0.2) = 0.22$, and solve for α numerically. Then find $\beta = 26 - \alpha$.

8.9. The nth moment equals $\frac{\sum_{j=0}^{n} \binom{n}{j} \frac{1}{(j+1)(n-j+1)}}{2^n}$.

8.11. Write the mean absolute deviation as $E(|X - \mu|) = c \int_0^{\frac{m}{m+n}} (\frac{m}{m+n} - x)x^{m-1}(1-x)^{n-1}dx + c\int_{\frac{m}{m+n}}^1 (x - \frac{m}{m+n})x^{m-1}(1-x)^{n-1}dx$, and then integrate term by term by expanding $(1-x)^{n-1}$. Here c is the normalizing constant of the Beta density.

8.12. $\alpha = 0.9825, \beta = 10.8548, P(X > .2) = 0.0862$. This problem requires numerical work.

8.14. $e^{-1/2}$.

8.15. 0.4008.

8.16. The density is $\frac{1}{2}e^{-(y-1)/2}, y \geq 1$.

8.17. Use the fact that $\sum_{i=1}^n X_i \sim G(n, 1)$ and then follow the same steps as in 8.16.

8.18. $1 - e^{-2}$.

8.21. The mean is $\prod_{i=1}^n m_i$ and the second moment is $\prod_{i=1}^n m_i(m_i + 2)$.

8.23. The number of centuries is $\frac{-\log(1-.5^{1/N})}{\log 2}$, where $N = 10^{25}$. This works out to approximately 83.6 centuries.

8.24. $P(X > 2\lambda) = e^{-2}$, which does not depend on λ.

8.25. (a) The expected profit is $E[(c_2 - c_1)\min(X, t)] - c_3 t$;

(b) The optimum value of t satisfies $1 - F(t) = \frac{c_3}{c_2 - c_1}$, provided $0 < \frac{c_3}{c_2 - c_1} < 1$.

8.28. The density function of $Y = e^{-X}$ is a standard exponential density.

8.29. The density function of $Y = \log\log\frac{1}{X}$ is $e^{-e^y}e^y, -\infty < y < \infty$.
8.30. The density function of $Y = \frac{\theta}{X}$ is $\alpha y^{\alpha-1}, 0 < y \leq 1$.
8.31. (a) 0.0465; (b) 0.0009; (c) this is the probability that $X > Y$ where X, Y are independent Poisson variables with means 4/3 and 1; see Theorem 6.13 in text for more details on how to compute it.
8.32. $\sqrt{\frac{s}{t}}$.
8.33. (a) The answer is yes; (b) the answer is no.
8.34. The distribution is $Bin(n, \frac{u}{t})$.

Chapter 9

9.1. 0.4772; 1; 0.1357; 0.5.
9.2. 2; 15.
9.3., The density of $Y = \frac{1}{Z}$ is $\frac{1}{y^2\sqrt{2\pi}}e^{-\frac{1}{2y^2}}, -\infty < y < \infty$. It is uniformly bounded.
9.4. $Z + 1$: all 1; $2Z - 3$: all -3; Z^3 : all zero.
9.5. The density of $Y = \phi(Z)$ is $\frac{2}{\sqrt{-\log(2\pi y^2)}}, 0 < y \leq \frac{1}{\sqrt{2\pi}}$.
9.7. 0.8230.
9.9. $\Phi(x) = \frac{1+\text{erf}(x/\sqrt{2})}{2}$.
9.10. $A : 6.68\%; B : 24.17\%; C : 38.3\%; D : 28.57\%; F : 2.28\%$.
9.12. If X denotes the diameter of a ball bearing, then we want $E(X|1.48 \leq X \leq 1.52)$ and $\text{Var}(X|1.48 \leq X \leq 1.52)$. These are the mean and the variance of a truncated distribution (section 4.12). For example, $E(X|1.48 \leq X \leq 1.52)$ equals $\frac{\int_{1.48}^{1.52} xf(x)dx}{P(1.48 \leq X \leq 1.52)}$. This is equal to 1.5 (why?). The variance calculation should be done by first calculating $E(X^2|1.48 \leq X \leq 1.52)$.
9.13. $Y = g(Z)$ has a mixed distribution. Its CDF is $F_Y(y) = 0(y < -a), = \Phi(-a)(y = -a), = \Phi(y)(-a < y < a), = 1(y \geq a)$.
9.14. 7.999 oz.
9.15. First prove that $E[\Phi(X)] = P(Z < X)$, where Z is an independent standard normal variable. Since $Z - X \sim N(-\mu, 1 + \sigma^2)$, $P(Z < X) = P(Z - X < 0) = \Phi(\frac{\mu}{\sqrt{1+\sigma^2}})$.
9.18. Median $= e^{\mu}$; mode $= e^{\mu-\sigma^2}$.
9.20. 0, 10, 0, 300.
9.21. 0.4778.

9.22. (a) 0.0037; (b) 0.3632.
9.23. 15367.
9.24. About 0.00315 at $x = 1.65$.

Chapter 10

10.1. Exact = 0.3770; normal approximation without continuity correction = 0.2643; normal approximation with continuity correction = 0.3759.
10.2. Exact = 0.1188; normal approximation = 0.1320.
10.3. 0.7887.
10.5. $\dot{0}.6124$. The first person has to make at least 442 correct predictions and the second person has to make at least 434 correct predictions. Find the probability of the intersection of these two events.
10.6. $P(Z \geq \frac{375.5-750\times.53}{\sqrt{750\times.53\times.47}}) = 0.9463$.
10.7. 0.0485.
10.9. This equals $P(\sum_{i=1}^{n}(X_i - X_i^2) > 0)$. Use the central limit theorem on these new variables $X_i - X_i^2$. The answers are: $n = 10 : 0.0339; n = 20 : 0.0049; n = 30 : 0.0008$.
10.10. 271 reservations can be allowed.
10.12. 0.0314.
10.13. 36 rolls should suffice.
10.15. First show that the mean and the variance of $\log X$ are -1 and 1. The final answers are 0.3228 for (a) and 0.0465 for (b).
10.16. 0.0058.
10.19. Formula for 99% confidence interval is $(X+3.315) \pm \sqrt{10.99 + 6.63X}$. Formula for 95% confidence interval is $(X + 1.92) \pm \sqrt{3.69 + 3.84X}$. For $X = 5$, 95% confidence interval is 6.92 ± 4.78, and 99% confidence interval is 8.315 ± 6.64.
10.22. (a) $.1n$; (b) $.99n$; (c) 0.8215.
10.23. Use the CLT for each of the totals scored by Tom and Sara and then find a normal approximation for the difference of these two totals. The final answer is 0.1271.
10.24. (a) $P(Z > 4) \approx 0$; (b) use normal approximation for Binomial with $n = 25, p = \int_{0.54}^{1} 6x(1-x)dx$; the final answer is 0.9649.
10.26. To find the third moment of S_n, expand $(X_1 + X_2 + \cdots + X_n)^3$ and

take expectations term by term. Then simplify.

10.27. The roundoff error on one transaction is uniformly distributed on $\{-50, -49, -48, \cdots, 0, \cdots, 49\}$. This gives $\mu = -0.5$. Now find σ^2, and then use a normal approximation to $1 - P(-5 \leq S_n \leq 5)$, with $n = 100$.

10.28. 0.2514.

10.29. 0.0002.

10.30. It is about 4.5 times more likely that you will get exactly 50 heads than that you will get more than sixty heads.

10.32. It is just a bit more likely that you will get exactly 20 sixes.

Chapter 11

11.1. The joint pmf of (X, Y) is $p(2,0) = 2/7, p(3,0) = p(3,2) = p(4,2) = p(6,1) = p(6,2) = 1/7$, and $p(x,y) = 0$ otherwise. $E(X) = 26/7; \mathrm{Var}(X) = 122/49; E(Y) = 1; \mathrm{Var}(Y) = 6/7; \rho_{X,Y} \approx 0.59$.

11.2. The joint pmf of (X, Y) is $p(0,4) = p(3,1) = p(4,4) = 1/16, p(1,2) = 1/4, p(2,0) = 3/8, p(3,2) = 3/16$, and $p(x,y) = 0$ otherwise. $E(Y) = 23/16$.

11.3. (a) $c = 1/36$; (b) They are independent; (c) $E(X) = E(Y) = 2, E(XY) = 4$.

11.4. (a) $c = 1/25$; (b) They are not independent; (c) $E(X) = 53/25; E(Y) = 67/25; E(XY) = 147/25$.

11.6. The joint pmf of (X, Y) is $p(1,3) = p(1,4) = p(1,5) = p(2,3) = p(2,5) = p(2,6) = p(3,4) = p(3,5) = p(3,7) = p(4,5) = p(4,6) = p(4,7) = 1/12$, and $p(x,y) = 0$ otherwise. X, Y are not independent.

11.7. The direct method of calculating $E(X|Y = y)$ is to use the formula $E(X|Y = y) = \frac{\sum_x xp(x,y)}{\sum_x p(x,y)}$. In this problem, $p(x,y) = \binom{13}{x}\binom{39}{13-x}\binom{13-x}{y}\binom{26+x}{13-y}, x, y, \geq 0, x + y \leq 13$. However, you can avoid the calculation by logically arguing how many clubs South (or any other player) should get, if y clubs have already been picked up by North. You can also logically see that $E(X|Y = 3)$ and $E(Y|X = 3)$ must be the same.

11.9. $E(Y) = 0.2$.

11.10. $\mathrm{Var}(X) = \frac{\lambda^2}{12} + \frac{\lambda}{2}$.

11.11. Use the fact that $P(X > Y)$ and $P(Y > X)$ are equal in this problem, and the fact that $P(X > Y) + P(Y > X) + P(X = Y) = 1$. Now find $P(X = Y)$ and then conclude that $P(X \geq Y) = \frac{1}{2-p}; P(X > Y) = \frac{1-p}{2-p}$.

11.13. pmf of $X+Y$ is $P(X+Y=2) = P(X+Y=8) = 0.04; P(X+Y=3) = P(X+Y=7) = 0.12; P(X+Y=4) = P(X+Y=6) = 0.21; P(X+Y=5) = 0.26$.

11.14. (a) They are not independent; (b) 0; (c) 0.

11.17. $X \sim Poi(\lambda+\eta); Y \sim Poi(\mu+\eta)$. X and Y are not independent. The joint pmf is found by simplifying the sum $P(X=x, Y=y) = \sum_{w=0}^{\min(x,y)} \frac{e^{-\lambda}\lambda^{x-w}}{(x-w)!} \frac{e^{-\mu}\mu^{y-w}}{(y-w)!} \frac{e^{-\eta}\eta^{w}}{w!}$.

11.18. (a) $3.5+y$; (b) $3.5y$; (c) $\frac{5369}{36}y^2$.

11.20. The joint pmf of (X,Y) is $p(2,0) = 1/2; p(3,0) = p(3,2) = p(4,0) = p(4,2) = 1/8$. $E(X|Y=0) = 3.5$. You can argue logically what $E(X|Y=2)$ should be.

11.22. You can prove this by using the result in problem 11.26. Show, by using problem 11.26, that $E(XY) = E_Y[E(XY|Y=y)] \geq E(Y)E(X)$, and so $\text{Cov}(X,Y) \geq 0$.

11.23. Use the two iterated expectation formulas $E(X) = E_Y[E(X|Y=y)]$ and $E(XY) = E_Y[E(XY|Y=y)]$.

11.24. Without loss of generality, you may assume that each of X, Y takes the values $0, 1$. Now try to represent the covariance between X and Y in terms of $P(X=0, Y=0) - P(X=0)P(Y=0)$. Finally show that if $P(X=0, Y=0) = P(X=0)P(Y=0)$, then that would force X, Y to be independent.

11.26. First show that for any two random variables U, V with finite variance, $\text{Cov}(U,V) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[P(U \geq u, V \geq v) - P(U \geq u)P(V \geq v)]dudv$. Apply this to $U = g(X), V = h(X)$.

11.27. The joint mgf is $(\frac{1}{6}e^{t_1} + \frac{1}{6}e^{t_2} + \frac{2}{3})^4$. The value of the covariance is $-\frac{1}{9}$.

11.28. The joint mgf is $\psi(t_1, t_2) = e^{-\lambda-\mu-\eta+\lambda e^{t_1}+\mu e^{t_2}+\eta e^{t_1+t_2}}$. Find $E(XY)$ by evaluating $\frac{\partial^2}{\partial t_1 \partial t_2}\psi$ at $(t_1, t_2) = (0, 0)$.

Chapter 12

12.1. (a) $c = 4$; (b) They are independent; (c) $f_X(x) = 2x, 0 \leq x \leq 1; f_Y(y) = 2y, 0 \leq y \leq 1, E(X) = E(Y) = 2/3; E(XY) = 4/9$.

12.2. (a) $c = 24$; (b) They are not independent; (c) $f_X(x) = 12x(1-x)^2, 0 \leq x \leq 1; f_Y(y) = 12y(1-y)^2, 0 \leq y \leq 1; E(X) = E(Y) = 2/5; E(XY) = 2/15$.

12.3. (a) $c = 1$; (b) They are not independent; (c) $f_X(x) = e^{-x}, x > 0; f_Y(y) = ye^{-y}, y > 0; E(X) = 1; E(Y) = 2$; (d) $E(X|Y = y) = y/2$; (e) $E(Y|X = x) = x + 1$.

12.5. (a) $c = \frac{3}{4\pi}$; (b) No; (c) $f_X(x) = \frac{3}{4}(1 - x^2), -1 \leq x \leq 1$; you can now find $f_Y(y), f_Z(z)$ by using a suitable symmetry in the formula for the joint density; (d) $E(X|Y = y) = 0$ for any y; (e) Similarly, $E(Y|X = x) = 0$ for any x; (f) 0.

12.6. $y^2 - y + \frac{1}{2}$.

12.8. $\frac{1}{2}$ (you can get the answer without doing any hard calculations).

12.9. $E(X\sqrt{X + Y}) = \frac{15}{8}\sqrt{\pi}$; use the fact that $\frac{X}{X+Y}$ and $X + Y$ are independent.

12.10. 3.

12.11. 18/55.

12.12. The joint distribution of (X, Y) does not have a density. The joint distribution is a distribution on the line segment $y = 2x, 0 \leq x \leq 1$. You can write the joint CDF of (X, Y) as $P(X \leq x, Y \leq y) = P(X \leq x, 2X \leq y) = P(X \leq x, X \leq \frac{y}{2})$ and simplify this last line.

12.14. $E(X_n) = \frac{1}{2^n} \to 0$.

12.16. 0.0294.

12.17. a, b satisfy $a(\sigma_1^2 + \rho\sigma_1\sigma_2) + b(\sigma_2^2 + \rho\sigma_1\sigma_2) = 0$.

12.18. 1/24.

12.20. 1/4. Note the similarity of this problem to Example 12.7 in the text.

12.21. 0.2248.

12.22. If we denote $X + Y = U, X - Y = V, Y = W$, then (U, V, W) does not have a joint density. The joint distribution of (U, V, W) is a distribution on the plane $w = \frac{u-v}{2}$. You can characterize the joint distribution of (U, V, W) by writing the joint distribution of (U, V) which is a bivariate normal, and the conditional distribution of W given $U = u, V = v$ is a one point distribution at $w = \frac{u-v}{2}$.

12.23. The correlation coefficient between X and X^2 is $\frac{\mu\sqrt{2}}{\sqrt{2\mu^2+\sigma^2}}$.

12.25. (a) $Y \sim N(1, 2)$; (b) $\frac{1}{\sqrt{2}}$; (c) $X|Y = y \sim N(\frac{y-1}{2}, \frac{1}{2})$.

12.26. The mean is 3/4, and the variance is 3/80.

12.27. (a) $\frac{3}{2\pi}\sqrt{1 - x^2 - y^2}, x^2 + y^2 \leq 1$; (b) $\frac{3}{4}(1 - x^2), -1 \leq x \leq 1$.

12.29. The mean residual life is $E(X|X > x) = \mu + \frac{\sigma\phi(\frac{x-\mu}{\sigma})}{1-\Phi(\frac{x-\mu}{\sigma})}$.
12.30. You have to find $E(Y|X > 140)$. This is different from $E(Y|X = 140)$. $E(Y|X > 140)$ equals $\frac{\int_{-\infty}^{\infty}\int_{140}^{\infty} yp(x,y)dxdy}{P(X>140)}$. $p(x, y)$ is a bivariate normal density, and you will be able to simplify the integral in the numerator. The denominator is easy because X is marginally normal.
12.31. (a) The densities of U, V, W are $3(1-u)^2(0 < u < 1), 6v(1-v)(0 < v < 1), 3w^2(0 < w < 1)$. (b) $\frac{U}{V} \sim U[0, 1]$ and $T = \frac{V}{W}$ has the density $2t(0 < t < 1)$. $\frac{U}{V}$ and $\frac{V}{W}$ are independent. (c) $E(\frac{U}{V}) = 1/2; E(\frac{V}{W}) = 2/3$.
12.33. U has an Exponential density with mean $\frac{1}{3}$. The densities of V, W are $6e^{-2v}(1-e^{-v})(v > 0)$ and $3e^{-w}(1-e^{-w})^2(w > 0)$. $T = W - U$ has the density $2e^{-t}(1-e^{-t})(t > 0)$.
12.34. (a) The epoch T of the last departure has the density $\frac{4}{\lambda}(e^{-t/\lambda} - e^{-2t/\lambda} - \frac{t}{\lambda}e^{-2t/\lambda})$; (b) 1/2; (c) The total time T spent by Mary at the post office has the density $\frac{2}{\lambda}e^{-t/\lambda}(1-e^{-t/\lambda})$.
12.36. (a) $(0.9)^n(1+\frac{n}{9})$; (b) Find the smallest n such that $2(0.99)^n - (0.98)^n \leq .01$. This gives $n = 527$.

Chapter 13

13.1. $U = XY$ has the density $f_U(u) = 2(1-u), 0 \leq u \leq 1$; $V = \frac{X}{Y}$ has the density $f_V(v) = \frac{2}{3}(0 \leq v \leq 1), = \frac{2}{3v^3}(v > 1)$.
13.2. $Z = X + Y$ has the density $f_Z(z) = z^2(0 \leq z \leq 1), = 2z - z^2(1 < z \leq 2)$. $V = X - Y$ has the density $f_V(v) = 1 - v^2(-1 \leq v \leq 0), = (1-v)^2(0 < v \leq 1)$. $W = |X - Y|$ has the density $f_W(w) = 2(1-w), 0 \leq w \leq 1$.
13.3. (a). No; (b) $c = 1/2$; (c) $Z = X + Y$ has the density $f_Z(z) = \frac{1}{2}z^2e^{-z}, z > 0$, which is a Gamma density.
13.4. (a) No; (b) $c = 8$; (c) $U = XY$ has the density $f_U(u) = \frac{-u\log u}{2}, 0 < u < 1$.
13.5. $T = NXY$ has a mixed distribution. T can be equal to zero with a positive probability; $P(T = 0) = \frac{1}{4}$. Conditional on $T > 0, T$ has a density. You have to find this by conditioning separately on $N = 1$ and $N = 2$, and take the mixture of the two densities that you will get for $1 \times (XY) = XY$ and $2 \times (XY) = 2XY$. To complete this problem, first show that $V = XY$ has the density $f_V(v) = -\log v, 0 < v < 1$, and then proceed as outlined

above.

13.6. $P(m = X) = 1 + \frac{2a^2 - 3a(b+c)}{6bc}$; $P(m = Z) = \frac{a(3b-a)}{6bc}$; $P(m = Y)$ can be found by subtraction.

13.8. $E(\frac{XY}{X^2+Y^2}) = E(\frac{XY}{\sqrt{X^2+Y^2}}) = 0$.

13.9. It is better to first find the CDF of $X^2 + Y^2$. This is easy for $t \leq 1$, but must be done carefully for $t > 1$. The CDF of $X^2 + Y^2$ is $P(X^2 + Y^2 \leq t) = \frac{\pi t}{4}(0 < t \leq 1)$, and $P(X^2 + Y^2 \leq t) = t(\frac{\pi}{4} - \arctan\sqrt{t-1}) + \sqrt{t-1}$ for $1 < t \leq 2$. Of course, for $t \geq 2, P(X^2 + Y^2 \leq t) = 1$.

To evaluate $E(\sqrt{X^2 + Y^2})$, apply its definition, and find the answer by numerical integration.

13.11. The point $P - Q$ has the planar coordinates $(U, V) = (X - Y, Z - W)$. First show that the joint density of (U, V) in polar coordinates is $\frac{2}{\pi^2}(\arccos(\frac{r}{2}) - \frac{r}{2}\sqrt{1 - \frac{r^2}{4}}), 0 \leq r \leq 2, -\pi < \theta < \pi$. This will lead to the density of r as $\frac{4}{\pi}r(\arccos(\frac{r}{2}) - \frac{r}{2}\sqrt{1 - \frac{r^2}{4}}), 0 \leq r \leq 2$. We need $E(r)$, which is $\frac{4}{\pi}\int_0^2 r^2(\arccos(\frac{r}{2}) - \frac{r}{2}\sqrt{1 - \frac{r^2}{4}})dr$. This integral equals $\frac{128}{45\pi} = 0.905415$, which is the needed average distance.

13.12. 1/6.

13.13. $P(\frac{X}{Y} < 1) = 1/4$; $P(X < Y) = 1/2$. They would have been the same if Y was a nonnegative random variable.

13.14. The answer is $\frac{2}{\pi}\arctan(\frac{\sigma}{\tau})$.

13.16. By using the general formula for the density of a product in the text, first show that the density of $U = XY$ is of the form $c_1c_2u^{\gamma-1}\int_u^1 x^{\alpha-\delta-\gamma}(1 - x)^{\beta-1}(x - u)^{\delta-1}dx$, where c_1, c_2 are the normalizing constants of the Beta densities for X, Y. Now simplify this expression in terms of the $2F1$ Hypergeometric function.

13.18. The density of $U = XY$ is $f_U(u) = \frac{2}{\pi}\frac{\log|u|}{u^2-1}, -\infty < u < \infty$.

13.20. This can be proved by simply using the Jacobian formula. You have to be a bit careful and take the cases $Y \leq \frac{1}{2}$ and $Y > \frac{1}{2}$ separately, and then put the two cases together.

13.21. The density of $W = X + Y + Z$ is $\frac{w^2 g(w)}{2}, w > 0$.

13.22. The required density is $\frac{3w^2}{(1+w)^4}, w > 0$.

13.24. The density of $Z = X + Y$ is $f_Z(z) = 1 - e^{-z}(0 \leq z \leq 1), = (e-1)e^{-z}, z > 1$.

13.25. The density of $Z = X + Y$ is $f_Z(z) = \Phi(\frac{z-\mu}{\sigma}) - \Phi(\frac{z-1-\mu}{\sigma})$.

13.27. Given $z \geq 0$, let n be such that $n \leq z < n+1$. Then the density of $Z = X + Y$ at z is $f_Z(z) = \frac{e^{-\lambda}\lambda^n}{n!}$.
13.28. (a) $c = \frac{1}{2\pi}$; (b) No; (c) r has the density $\frac{r}{(1+r^2)^{3/2}}, r > 0, \theta \sim U[-\pi, \pi]$, and r, θ are independently distributed; (d) $1 - \frac{1}{\sqrt{2}}$.
13.30. The correlation is 0.263038.
13.31. The correlation is 0.
13.32. The density of $Z = X + Y$ is $f_Z(z) = \frac{e^{-z/\mu} - e^{-z/\lambda}}{\mu - \lambda}, z > 0$; the density of $W = X - Y$ is $f_W(w) = \frac{1}{\lambda+\mu} e^{w/\mu} (w < 0)$, and $= \frac{1}{\lambda+\mu} e^{-w/\lambda} (w > 0)$.
13.35. If $X \sim U[0, 1]$, then $n_1, n_2, \cdots$ are each uniform on $\{0, 1, 2, \cdots, 9\}$, and they are independent.
13.36. The integer part of X and the fractional part of X are independent in this case. The integer part has the pmf $P(\lfloor X \rfloor = n) = e^{-n}(1 - e^{-1}), n = 0, 1, 2, \cdots$, and the fractional part has the CDF $P(\{X\} \leq y) = \frac{e^y - 1}{e^{y-1}(e-1)}, 0 \leq y \leq 1$. Note the interesting fact that the integer part therefore has a Geometric distribution.
13.38. The density of $R = \sqrt{X_1^2 + X_2^2 + \cdots + X_n^2}$ is $f_R(r) = \frac{4\Gamma(\frac{n+3}{2})}{\sqrt{\pi}\Gamma(\frac{n}{2})} \frac{r^{n-1}}{(1+r^2)^{(n+3)/2}}, r > 0$. You can find the density of $X_1^2 + X_2^2 + \cdots + X_n^2$ from this by a simple transformation.

Hints and Answers to Supplementary Problems

Appendix I: Word Problems

I.1. 17576; 18252.
I.2. 0.01; 0.72; 0.27.
I.3. 11/16.
I.4. 1/32.
I.5. 1/6; 1/3; 1/2.
I.7. 0.988.
I.8. The winning probabilities of A, B, C are 36/91, 30/91, 25/91 respectively.
I.11. 0.0435.
I.12. 9/29.
I.14. 0.1054.
I.16. For with replacement sampling, the probability is 1/9; for without replacement sampling, the probability is 3/55.
I.17. $\binom{4}{1}\frac{\binom{39}{13}}{\binom{52}{13}} - \binom{4}{2}\frac{\binom{26}{13}}{\binom{52}{13}} + \binom{4}{3}\frac{\binom{13}{13}}{\binom{52}{13}}$.
I.19. $\sum_{x=5}^{10} \frac{\binom{39}{x-1}\binom{13}{1}}{\binom{52}{x}}$.
I.20. $\frac{\binom{6}{4}}{\sum_{x=3}^{6}\binom{6}{x}}$.
I.21. 0.30199.
I.23. $\frac{4!}{4^4}$.
I.26. $\sum_{x=0}^{10}\left[\frac{\binom{25}{x}\binom{25}{10-x}}{\binom{50}{10}}\right]^2$.
I.28. $1-(0.6)^5$.
I.29. 1/5.
I.30. Denoting the selection of a green ball by G and that of a white ball by W, the favorable sample points are $GGGGG, GGGWG, GWGGG, GWGWG$. Compute the probability of each of these sample points and add.
I.31. 1/3.
I.33. $0.8740; 0.5282; 0.559$.
I.35. (a) 2; (b) all numbers in $[1, 2]$ are medians of the distribution; (c) First prove that $\sum_{n=1}^{\infty}\frac{n(n-1)}{2^n} = 4$.
I.36. $\mathrm{Var}(|X|) = E(X^2) - [E(|X|)]^2 \leq E(X^2) - |E(X)|^2$ (since $E(|X|) \geq$

$|E(X)|) = \text{Var}(X)$.

I.38. This is the negative hypergeometric distribution (see Exercise 6.25). (a) To find the mass function, show that $P(X > x) = \frac{\binom{39}{x}}{\binom{52}{x}}, x = 1, 2, \cdots, 39$, and then find the mass function by subtraction; (b) By the tailsum formula, $E(X) = 1 + \sum_{x=1}^{39} \frac{\binom{39}{x}}{\binom{52}{x}} = 53/14 = 3.786$.

I.40. Consider a two valued random variable X with the mass function $P(X = \mu - 0.005) = 1 - \delta, P(X = N) = \delta$. If $\delta = \frac{0.005}{N+0.005-\mu}$, then X has mean μ. Now take N to be sufficiently large.

I.41. Since $E(X) = \mu = E(X^2)$, and $E(X^2) \geq (E(X))^2$ for any random variable X, it follows that $\mu \geq \mu^2$, i.e., $0 \leq \mu \leq 1$. Therefore, $\text{Var}(X) = E(X^2) - (E(X))^2 = \mu - \mu^2 = \mu(1 - \mu) \leq \frac{1}{4}$.

I.44. (a) $E(X) = E(Y) = 4/3$; (b) $\text{Var}(X) = \text{Var}(Y) = 8/9$; (c) $P(Z = 0) = 4/9; P(Z = 1) = 14/27; P(Z = 2) = 1/27$; (d) $E(Z) = 16/27$.

I.45. The variance is 11.67.

I.47. Mean $= 4.405$; variance $= 3.63$.

I.49. For example, take X such that $P(X = 0) = \frac{1}{10^4+1}, P(X = 100.01) = \frac{10^4}{10^4+1}$.

I.50. $4\frac{\binom{36}{13}}{\binom{52}{13}}$.

I.51. (a) 10; (b) Show that $P(N > n) = \frac{10\times 9\times\cdots\times(11-n)}{10^n}, n = 1, 2, \cdots, 10$. Then, by the tailsum formula, $E(N) = 2 + \sum_{n=2}^{10} P(N > n)$.

I.52. $91/6$.

I.54. $1 - \log 2$.

I.55. The mgf equals $\psi(t) = \frac{1+e^t+e^{2t}+e^{3t}+e^{4t}+e^{5t}}{5e^{2t}}$. The mean is zero.

I.56. $E(Y) = \frac{2\binom{n}{2}+4\binom{n}{4}+\cdots}{2^n} = \frac{n}{4}$; the mgf of Y equals $\psi(t) = \frac{1}{2}+\frac{1+\binom{n}{2}e^{2t}+\binom{n}{4}e^{4t}+\cdots}{2^n}$.

I.57. The mgf of XY is $1 - p + pe^{\lambda(e^t-1)}$.

I.59. The mgf is $\prod_{i=1}^n (1 - p_i + p_i e^t)$; the variance is $\sum_{i=1}^n p_i(1 - p_i)$.

I.60. X takes the values ± 1 with probability $\frac{1}{2}$ each,

I.62. (a) The mgf is $e^{-\lambda} \sum_{x=0}^n \frac{(\lambda e^t)^x}{x!} + P(X > n)$; (b) The limit equals the mgf of X itself, i.e., $e^{\lambda(e^t-1)}$.

I.64. The nth factorial moment is λ^n.

I.65. (a) $c = \frac{1}{8}$; (b) $p(0) = 1/8; p(1) = 1/2; p(2) = 1/4; p(3) = 1/8$; (c) $11/8$.

I.66. $e^{(s-1)\lambda}$.

I.68. Yes, it is possible. Take a random variable X with the stated property and take $Y = -X$.

I.70. .1662

I.71. For sending one parcel, the expected cost is \$9.50. For sending two parcels, the expected cost is \$11.50. It is better to send one parcel.

I.73. The distribution of X is a binomial with parameters 3 and $\frac{1}{36}$. Therefore, the mean is $\frac{3}{36}$ and the variance is $3\frac{1}{36}\frac{35}{36}$.

I.75. $e^{-25/6}$.

I.77. $\frac{p}{\theta}$.

I.79. (a) 0.0835; (b) 0.0207; (c) $k = 15$ in each case.

I.80. (a) $\frac{\binom{39}{13}}{\binom{52}{13}}$; (b) 0.3664; (c) 0.3658.

I.82. $P(\max(X,Y) \leq n) = 1 - .4^n - .5^n + .2^n, n \geq 0$. By using the tailsum formula now, $E(\max(X,Y)) = \sum_{n=0}^{\infty}(.4^n + .5^n - .2^n) = \frac{29}{12} = 2.42$.

I.83. The mgf of $X - Y$ is $e^{-\lambda-\mu+\lambda e^t+\mu e^{-t}}$.

I.85. The first four moments are $2, 6, 22, 94$.

I.86. The first four moments are $5, 27.5, 162.5, 1017.5$.

I.87. 0.

I.88. Use the tailsum method to find the mean.

I.90. Suppose the couple has N children. First prove that $P(N > n) = \sum_{x=n-r+1}^{r-1} \binom{n}{x} p^x (1-p)^{n-x}, r \leq n \leq 2r-2$. Then find the pmf of N by subtraction.

I.91. $\frac{2587}{630} = 4.11$.

I.92. No; because by problem I.93, $\text{Var}(XY) > \text{Var}(X)\text{Var}(Y) = E(X)E(Y) = E(XY)$.

I.94. X must be ± 1 with probability $\frac{1}{2}$ each.

I.96. 1.06.

I.97. (a) The density is $f(x) = 0(x < 1), = (x-1)/3(1 \leq x \leq 2), = 1/3(2 \leq x \leq 4), = (5-x)/3(4 \leq x \leq 5), = 0(x > 5)$; (b) The CDF is $F(x) = 0(x \leq 1), = (x-1)^2/6(1 < x \leq 2), = x/3 - 1/2(2 < x \leq 4), = 5x/3 - (x^2+19)/6(4 < x < 5), = 1(x \geq 5)$; (c) 3; (d) $\frac{32}{3}$.

I.98. (a) $c = 4$; (b) 0; 0.75; 0.75; $P(C|A \cap B) = \frac{0}{0}$ and is undefined.

I.100. $F(x) = 0(x < 0), = 0.9(x = 0), = 1 - \frac{1}{10}e^{-\frac{x}{2}}(x > 0)$.

I.102. (a) 0; (b) 1; (c) prove that $x^3 - x^2 - x - 2 > 0$ if $x > 2$. So the

required probability is $P(X > 2)$; (d) $\frac{1}{2}(1 + e^{-3})$; (e) 0.
I.103. The density of $Y = e^{-X^2}$ is $f(y) = \frac{1}{2y\sqrt{-\log y}}(\frac{1}{e} \leq y < 1)$, and 0 otherwise.
I.105. The density of $Y = g(X)$ is $f(y) = e^{-y} + \frac{e^{-\frac{1}{y}}}{y^2}, 0 < y < 1$ and 0 otherwise.
I.107. Y has the CDF $F(y) = \frac{1+y}{2}, (-1 \leq y \leq 1), = 0(y < -1), = 1(y > 1)$. In other words, Y has the same distribution as Z.
I.109. (a) σ equals 1.53; (b) 6.96; (c) 0.9995.
I.110. The shortest interval with probability $\geq .5$ under each of these three distributions is $[-1, 1]$.
I.111. $\sqrt{.6}$.
I.114. Mean equals $\frac{1}{n}$, median equals $\frac{\log 2}{n}$, and variance equals $\frac{1}{n^2}$.
I.115. $\frac{1}{3}$.
I.116. 4.
I.117. $-\log(\frac{1}{2} - \frac{1}{\pi}\arctan X)$ is one such transformation.
I.118. (a) $c = \frac{1}{\pi}$; (b) $\frac{\pi^2-4}{\pi}$.
I.120. 2 square inches.
I.121. $2\sqrt{2\pi}$.
I.123. $n \geq \frac{\log .1}{\log(\Phi(4))} \approx 72702$.
I.124. $a < \frac{1}{2}$.
I.125. The function has a minima of zero and a maxima of 1.
I.127. Hazard rates with a bathtub shape can be generated by using densities of the form $cx^{d-1}e^{-x^p}, x > 0, p, d > 0$. Of course, this is not the only possible way to generate bathtub hazard rates.
I.129. $(0.9)^{10}$.
I.130. $(0.9544)^{10}$.
I.131. The deciles are $e^{-1.28}, e^{-0.84}, e^{-0.525}, e^{-0.253}, 1, e^{0.253}, e^{0.525}, e^{0.84}, e^{1.28}$.
I.132. The name is lognormal.
I.133. No, because the mean μ would be zero, and then the three percentile values become inconsistent.
I.135. For a 90% confidence interval, n approximately $271\sigma^2$; for a 95% confidence interval, n approximately $384\sigma^2$; for a 99% confidence interval, n approximately $663\sigma^2$.
I.136. 0.6748.

I.137. For Z^5, the mean, median, and mode are all zero; for $|Z|$, the mean is $\sqrt{\frac{2}{\pi}}$, the median is 0.675, and the mode is zero; for $|Z-1|$, the mean is $\sqrt{\frac{2}{\pi e}}+2\Phi(1)-1$, the median is 1.05, and the mode is zero.
I.139. The distribution is approximately a normal distribution with mean $\frac{50}{6}$ and variance $\frac{550}{36}$.
I.141. n approximately 3393.
I.143. 0.0793.
I.144. A normal approximation cannot be used here, because this density does not have a finite mean.
I.145. 0.0031.
I.149. The conditional expectation of the number of heads in the last 15 tosses given that the number of heads in the first 15 tosses is x is given by the formula $\frac{5}{2}+\frac{2}{3}x$.
I.150. By using the iterated variance formula, you can easily prove that $\text{Var}(Y) \geq \text{Var}(X)$.
I.151. (a) Each of X, Y, Z takes the values ± 1 with probability $\frac{1}{2}$ each; (b) each of these joint pmfs assigns probability $\frac{1}{4}$ to the four points $(\pm 1, \pm 1)$; (c) each correlation is zero; (d) $\frac{1}{2}$.
I.152. (a) The variance is $\frac{3n}{4}$; (b) the limit is zero by Chebyshev's inequality.
I. 153. The percentages of A, B, C, D, and F would be 2.28%, 13.59%, 68.26%, 13.59% and 2.28%. So the required probability is $\frac{40!}{(8!)^5}(0.0228)^8(0.1359)^8(0.6826)^8(0.1359)^8(0.0228)^8$.
I.155. 0.
I.156. (a) $2(1-2^{-k})$; (b) $\min(x,k)$; (c) The covariance between X and $\min(X,k)$ is $2-2(k+1)2^{-k}$. The variance of $\min(X,k)$ is $2+2^{1-k}-4^{1-k}-4k2^{-k}$. Using these, the correlation between X and $\min(X,k)$ is $\frac{2^k-k-1}{\sqrt{4^k-(2k-1)2^k-2}}$. This converges to 1 when $k \to \infty$.
I.157. (a) The expected values are $\frac{N+1}{2}$; (b) $\frac{\frac{N(N+1}{2}-n}{N-1}$; (c) Show that the covariance between the two numbers drawn is $-\frac{N+1}{12}$. The correlation is $-\frac{1}{N-1}$; (f) 0.
I.159. (a) $\frac{7}{12}$; (b) $\frac{161}{216}$; (c) $\frac{91}{216}$.
I.160. $E(X_1X_2X_3)=n(n-1)(n-2)p_1p_2p_3$.
I.162. For a general k, the conditional expectation is $\frac{4-k}{3}$.
I.164. (a) $E(X|Y=y)=\frac{3}{4}(1-y)$; (b) $E(Y|X=x)=\frac{3}{4}(1-x)$; (c)

$E(XY) = \frac{9}{56}$; (d) $E(X^2Y^2) = \frac{1}{35}$.
I.165. (a) $\frac{1}{2}$; (b) 0; (c) $-\phi(c)\Phi(c)$; (d) $\frac{1}{4} - \frac{1}{2\pi}\arcsin(\frac{3}{\sqrt{10}})$.
I.166. The density of $Z = X - Y$ is $f_Z(z) = 1 - |z|, -1 \leq z \leq 1$.
I.167. A necessary and sufficient condition that you can make a triangle with three segments of lengths a, b, c is that the sum of any two of a, b, c is at least as large as the third one. Using this, you can show that the required probability in this problem is $2\log 2 - 1$.
I.168. $\frac{1}{2}$.
I.169. The joint density of $U = X, V = XY, W = XYZ$ is $f_{U,V,W}(u, v, w) = \frac{e^{-u-\frac{v}{u}-\frac{w}{v}}}{uv}, u, v, w > 0$.
I.172. (a) Show that $\text{Cov}(X_1 + X_2, X_1 - X_2) = 0$, and that it follows from this that $X_1 + X_2, X_1 - X_2$ are independent; (b) since $X_1 + X_2, X_1 - X_2$ are independent, any function of $X_1 + X_2$ and any function of $X_1 - X_2$ are independent; (c) write $X_1^2 + X_2^2$ and $\frac{X_1}{X_2}$ in terms of the polar coordinates, from which it will follow that they are independent.
I.173. (a) $E(XY) = \frac{1}{2}, \text{Var}(XY) = \frac{5}{4}$; (b) $E(X^2Y) = 0$; (c) 0; (d) $c = -1$.
I.174. 0.0367.
I.176. $\frac{1}{2}$.
I.177. $\frac{1}{3}$.
I.178. Yes, the expectation is finite and equals $\sqrt{2}$.
I.180. (a) A bound for the density of $Z = X + Y$ is $f_Z(z) \leq \frac{1}{\sqrt{2\pi}}$ for all z; (b) no, the density of XY may not be bounded. Think of a random variable Y with a density converging to ∞ very sharply at $y = 0$.
I.182. $\frac{4}{9}$.
I.183. $\frac{\theta(1-p)}{p+\theta(1-p)}$.
I.184. The mean equals $10(1 - e^{-10})$ by applying the iterated expectation formula; the variance has to be found by applying the iterated variance formula, and the terms in the formula have to be found numerically. The variance is approximately 21.90.
I.185. 22.

Appendix I: True-False Problems

1. T.
4. T.
6. F.
8. T.
9. F.
11. F.
13. F.
14. T.
15. F.
16. T.
18. F.
19. F.
22. T.
23. T.
25. T.
27. T.
28. F.
30. T.
32. F.
33. F.
35. T.
36. T.
39. T.
40. T.
42. T.
45. T.
46. T.
47. T.
49. T.
50. F.
51. T.
52. F.
53. T.

55. F.
57. T.
59. F.
61. T.
63. T.
64. T.
65. T.
66. T.
68. T.
70. T.
71. F.
72. T.
73. T.
76. T.
77. T.
78. T.
79. F.
81. T.
82. T.
84. F.
86. F.
87. T.
88. F.
90. T.
91. F.
94. F.
95. T.
97. T.
99. F.
100. T.
101. T (the expected value is 2.49).
103. T.
104. T.
106. F.

107. T.
109. T.
110. F.
112. T.
113. T.
114. F (ρ can also be -0.8).
116. T.
118. T.
119. F.
121. T