Probability and Statistics

Solution sheet 10

Solution 10.1 Let θ follows a continuous distribution,

(a) let $a \leq q$,

$$\begin{aligned}\mathbb{E}[L(\theta,a)-L(\theta,q)] =&\mathbb{E}[(L(\theta,a)-L(\theta,q))(\mathbb{1}_{\{\theta\leq a\}}+\mathbb{1}_{\{a<\theta<q\}}+\mathbb{1}_{\{q\leq\theta\}})]\\ =&\mathbb{E}[c(a-\theta-q+\theta)\mathbb{1}_{\{\theta\leq a\}}]+\mathbb{E}[(\theta-a-c(q-\theta))\mathbb{1}_{\{a<\theta<q\}}]\\ &+\mathbb{E}[(\theta-a-\theta+q)\mathbb{1}_{\{q\leq\theta\}}]\\ =&c(a-q)\mathbb{P}(\theta\leq a)+(1+c)\mathbb{E}(\theta\mathbb{1}_{\{a<\theta<q\}})-(a+cq)\mathbb{P}(a<\theta<q)\\ &+(q-a)\mathbb{P}(q\leq\theta)\\ \geq&(q-a)[\mathbb{P}(q\leq\theta)-c\mathbb{P}(\theta\leq a)]+a(1+c)\mathbb{P}(a<\theta<q)\\ &-(a+cq)\mathbb{P}(a<\theta<q)\\ =&(q-a)[\mathbb{P}(q\leq\theta)-c\mathbb{P}(\theta\leq q)].\end{aligned}$$

Similarly we have

$$\mathbb{E}[L(\theta,a)-L(\theta,q)]\leq(q-a)[\mathbb{P}(a\leq\theta)-c\mathbb{P}(\theta\leq a)].$$

(b) Let q be a $1/(1+c)$-quantile of θ. Then

$$\mathbb{P}(\theta\leq q)=1/(1+c)\text{ and }\mathbb{P}(\theta\geq q)=c/(1+c).$$

For all $a \leq q$, by the first inequality,

$$\mathbb{E}(L(\theta,a)-L(\theta,q))\geq 0.$$

For all $a \geq q$, by the second inequality,

$$\mathbb{E}\left[L(\theta,a)-L(\theta,q)\right]\geq(q-a)\left(\mathbb{P}[q\leq\theta]-c\mathbb{P}[\theta\leq q]\right)=0.$$

Thus q minimizes $\mathbb{E}[L(\theta,a)]$ among all $a\in\mathbb{R}$, it is the Bayes estimator with loss function L.

Solution 10.2

(a) Recall that X follows the Poisson distribution with mean θ then

$$\mathbb{P}[X=k]=e^{-\theta}\frac{\theta^k}{k!}.$$

And θ has the probability density function

$$f_\theta(x)=\frac{x^{3-1}e^{-x}}{\Gamma(3)}=x^2e^{-x}/2.$$

The conditional probability density function of θ is obtained as before (compare Exercise 8.5),

$$\begin{aligned}f_{\theta|2,2,6,0,3}(x)&=\frac{x^2e^{-x}/2\cdot e^{-5x}x^{2+2+6+0+3}/(2!2!6!0!3!)}{\int_0^\infty s^2e^{-s}/2\cdot e^{-5s}s^{2+2+6+0+3}/(2!2!6!0!3!)ds}\\ &=\frac{e^{-6x}x^{15}}{\int_0^\infty e^{-6s}s^{15}ds}\\ &=e^{-6x}x^{15}\frac{6^{16}}{\Gamma(16)}:=\xi(x|2,2,6,0,3)\end{aligned}$$

Remark also that the posterior distribution of θ is the Gamma distribution with parameter $(\alpha = 16, \beta = 6)$.

(b) The squared error loss function is

$$L(\theta, a) = (\theta - a)^2.$$

The Bayes estimator of θ is the a which minimizes

$$\mathbb{E}[L(\theta, a)|\text{observation}] = \int L(x, a)\xi(x|\text{observation})dx,$$

where $\xi(\theta|\text{observation})$ is the posterior probability density function of θ given the observation. We have computed the k-th moments of Gamma distribution X with parameter α and β in previous series, in particular

$$\mathbb{E}(X) = \frac{\alpha}{\beta}, \quad \mathbb{E}(X^2) = \frac{\alpha(\alpha+1)}{\beta^2}.$$

Hence

$$\begin{aligned}\mathbb{E}[(\theta - a)^2|2, 2, 6, 0, 3] &= \mathbb{E}[\theta^2|2, 2, 6, 0, 3] - 2a\mathbb{E}[\theta|2, 2, 6, 0, 3] + a^2 \\ &= \frac{17 \times 16}{6^2} - 2a\frac{16}{6} + a^2 \\ &= \left(a - \frac{8}{3}\right)^2 + \frac{4}{9}.\end{aligned}$$

The Bayes estimate for the observation $2, 2, 6, 0, 3$ is $8/3$.

Solution 10.3 First we calculate the joint probability density function of $X_1, \cdots, X_n, \theta$:

$$\begin{aligned}f_{X_1,\cdots,X_n,\theta}(x_1, \cdots, x_n, x) &= x^y(1-x)^{n-y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1} \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha+y-1}(1-x)^{\beta+n-y-1}\end{aligned}$$

where $y = x_1 + \cdots + x_n$. So that the marginal probability density function of $X_1, \cdots, X_n$ at $(x_1, \cdots, x_n)$ is

$$\int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha+y-1}(1-x)^{\beta+n-y-1}dx = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha+y)\Gamma(\beta+n-y)}{\Gamma(\alpha+\beta+n)}.$$

Thus the conditional probability density function of θ given $x_1, \cdots, x_n$ is

$$f_{\theta|x_1,\cdots,x_n}(x) = \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+y)\Gamma(\beta+n-y)}x^{\alpha+y-1}(1-x)^{\beta+n-y-1},$$

which is the Beta distribution with parameters $\alpha + y$ and $\beta + n - y$.

Solution 10.4

(a) The integral of ξ is

$$\begin{aligned}\int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{-(\alpha+1)}e^{-\beta/\theta}d\theta =& \frac{\beta^\alpha}{\Gamma(\alpha)}\int_\infty^0 -\frac{\beta}{y^2}\beta^{-(\alpha+1)}y^{\alpha+1}e^{-y}dy \\ =& \frac{1}{\Gamma(\alpha)}\int_0^\infty y^{\alpha-1}e^{-y}dy \\ =& 1\end{aligned}$$

(b) Let Θ be a random variable following the inverse gamma distribution with parameter α and β, and let X be a random variable such that the conditional probability density function given $\Theta = \theta$ is

$$f_{X|\theta}(x) = \frac{1}{\sqrt{2\pi\theta}} e^{-(x-\mu)^2/2\theta}.$$

The conditional probability density function of Θ given $X = x$ is

$$\begin{aligned} f_{\Theta|x}(\theta) &= \frac{\xi(\theta) f_{X|\theta}(x)}{\int_0^\infty \xi(\theta) f_{X|\theta}(x) d\theta} \\ &= \frac{\xi(\theta) f_{X|\theta}(x)}{\frac{\beta^\alpha}{\Gamma(\alpha)\sqrt{2\pi}} \int_0^\infty \theta^{-(\alpha+1/2)-1} e^{-(\beta+(x-\mu)^2/2)/\theta} d\theta} \\ &= \frac{\frac{\beta^\alpha}{\Gamma(\alpha)\sqrt{2\pi}} \theta^{-(\alpha+1/2)-1} e^{-(\beta+(x-\mu)^2/2)/\theta}}{\frac{\beta^\alpha}{\Gamma(\alpha)\sqrt{2\pi}} \frac{\Gamma(\alpha+1/2)}{[\beta+(x-\mu)^2/2]^{\alpha+1/2}}} \\ &= \frac{(\beta')^{\alpha'}}{\Gamma(\alpha')} \theta^{-\alpha'-1} e^{-\beta'/\theta}. \end{aligned}$$

Where we set $\alpha' = \alpha + 1/2$ and $\beta' = \beta + (x-\mu)^2/2$. The conditional distribution of Θ given $X = x$ is an inverse Gamma distribution with parameter α' and β'. Thus the family of inverse Gamma distribution is a family of prior distributions for samples from a normal distribution with a known mean μ and an unknown value of the variance.

Solution 10.5 At time $t = 2$, X, Y, Z are independently and identically normally distributed with mean 0 and variance $2\sigma^2$. Let $\|x, y, z\|$ denote the euclidean distance from (x, y, z) to 0,

$$\mathbb{P}(\|X, Y, Z\| \le 4\sigma) = \mathbb{P}((X^2 + Y^2 + Z^2)/2\sigma^2 \le 8).$$

Since $X^2 + Y^2 + Z^2 = 2\sigma^2[(X/\sqrt{2}\sigma)^2 + (Y/\sqrt{2}\sigma)^2 + (Z/\sqrt{2}\sigma)^2]$, we have

$$\frac{X^2 + Y^2 + Z^2}{2\sigma^2} \sim \chi^2(3).$$

The distribution of $\chi^2(3)$ is a Gamma distribution with parameter $(3/2, 1/2)$, with probability density function

$$f(x) = \frac{(1/2)^{3/2}}{\Gamma(3/2)} x^{1/2} e^{-x/2}.$$

We find that the probability of the particle to be in the ball of radius 4σ around 0 is the probability of a $\chi^2(3)$-distributed random variable being less than 8:

$$\mathbb{P}[(X^2 + Y^2 + Z^2)/2\sigma^2 \le 8] = \frac{(1/2)^{3/2}}{\Gamma(3/2)} \int_0^8 x^{1/2} e^{-x/2} dx.$$

Solution 10.6

(a) First we have to prove that U is measurable. For this we observe that $U^{(m)} := \sum_{n=1}^m U_n 2^{-n}$ is measurable as a finite sum of measurable functions and we have the pointwise convergence

$$U_n \to U.$$

Second we have to understand the measure that U produces on $\mathbb{R}$. For this, it is enough to show that the measure induced by U coincides with the uniform measure on the intervals of the form

$$\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right],$$

for $k \in \mathbb{N} \in [0, 2^n - 1]$. This is because they generate the Borel σ-algebra.
Note that

$$\mathbb{P}[(\exists n \in \mathbb{N})(\forall m \geq n) X_m = 1] = 0,$$

thanks to Borel-Cantelli Lemma. So we can work in

$$\tilde{\Omega} := \Omega \backslash \{\omega \in \Omega : (\exists n \in \mathbb{N})(\forall m \geq n) X_m = 1\},$$

i.e. our probability space is $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}})$ where $\tilde{\mathcal{A}} = \mathcal{A} \mid_{\tilde{\Omega}}$ and $\tilde{\mathbb{P}} := \mathbb{P} \mid_{\tilde{\Omega}}$. Now we have that if $k = \sum_{i=0}^{n-1} k_i 2^i \in \{0, 1, .., 2^n - 1\}$:

$$\begin{aligned}
\mathbb{P}\left[U \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right] &= \mathbb{P}\left[\bigcap_{i=0}^{2^n-1} \{U_{i+1} = k_{n-i}\}\right] \\
&= \prod_{i=0}^{2^n-1} \mathbb{P}[U_{i+1} = k_{n-i}] \\
&= 2^{-n}.
\end{aligned}$$

That is the probability of a uniform random variable to be in $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$.

(b) Take $(U_n)_{n \in \mathbb{N}}$ Bernoulli with parameter $p = \frac{1}{2}$ i.i.d. Thanks to part (a) we have that $U := \sum_{n=1}^{n} U_n 2^{-n}$ is uniform distributed and it is normal if and only if

$$\frac{\sum_{k=1}^{n} \mathbb{1}_{\{U_k = 1\}}}{n} \to \frac{1}{2}.$$

Then:

$$\mathbb{P}[U \text{ is normal}] = \mathbb{P}\left[\frac{\sum_{k=1}^{n} \mathbb{1}_{\{U_k=1\}}}{n} \to \frac{1}{2}\right] = 1.$$

Where in the last equality we have used the strong law of large numbers.