Probability and Statistics

Solution sheet 11

Solution 11.1 The Poisson distribution with mean λ has variance λ, thus the standard deviation σ is $\sqrt{\lambda}$. Given the parameter λ, the probability density function of the Poisson distribution is

$$\mathbb{P}[X = k|\sigma] = e^{-\sigma^2}\frac{\sigma^{2k}}{k!}.$$

Hence the M.L.E. is the value σ which maximizes

$$f(\sigma) = \prod_{i=1}^{n} e^{-\sigma^2}\frac{\sigma^{2X_i}}{X_i!} = \frac{(e^{-\sigma^2}\sigma^{2\bar{X}})^n}{X_1!\dots X_n!}.$$

Where $\bar{X} = (X_1 + \cdots + X_n)/n$. We need to find the σ which maximizes

$$g(\sigma) = e^{-\sigma^2}\sigma^{2\bar{X}} = \exp(-\sigma^2 + 2\bar{X}\ln(\sigma)).$$

$$g'(\lambda) = (-2\sigma + 2\bar{X}/\sigma)g(\sigma).$$

The maximum of g is reached when $\sigma = \sqrt{\bar{X}}$. Thus the M.L.E. of the standard deviation is $\sqrt{\bar{X}}$.

Solution 11.2 Let

$$L(\theta) := \prod_{i=1}^{n} \theta X_i^{\theta-1} = \theta^n \left(\prod_{i=1}^{n} X_i\right)^{\theta-1}.$$

The derivative of $\ln(L)(\theta)$ is:

$$\left[n\ln(\theta) + (\theta - 1)\ln\left(\prod_{i=1}^{n} X_i\right)\right]' = \frac{n}{\theta} + \ln\left(\prod_{i=1}^{n} X_i\right).$$

Thus

$$\theta_0 = -\frac{n}{\ln\left(\prod_{i=1}^{n} X_i\right)} = -\frac{1}{\overline{\ln(X)}},$$

where $\overline{\ln(X)} = \frac{1}{n}\sum_{i=1}^{n}\ln(X_i)$, is a critical point of L. For $\theta < \theta_0$, $\ln(L)$ is increasing and for $\theta > \theta_0$, $\ln(L)$ is decreasing. Thus θ_0 is the global maximum, and is the M.L.E. of θ.

Solution 11.3 Define X the amount of marked fishes we fished. If there are N fishes in the lake, the probability of $X = 3$ is given by

$$\begin{aligned}\mathbb{P}_N[X = 3] &= \frac{\binom{5}{3}\binom{N-5}{8}}{\binom{N}{11}}\mathbb{1}_{\{N\geq 13\}} \\ &= \frac{5!(N-5)!11!(N-11)!}{3!2!8!(N-13)!N!}\mathbb{1}_{\{N\geq 13\}} := g(N).\end{aligned}$$

We have to find $N_{\max} \in \mathbb{N}$ so that $g(N_{\max}) = \sup_{N\in\mathbb{N}} g(N)$. We have that for $N \geq 13$

$$\begin{aligned}\frac{g(N)}{g(N+1)} - 1 &= \frac{(N-12)(N+1)}{(N-4)(N-10)} - 1 \\ &= \frac{3(N - 17,333\dots)}{(N-4)(N-10)},\end{aligned}$$

thus,

$$\frac{g(N)}{g(N+1)} \begin{cases} \leq 1 & \text{if } N \leq 17, \\ \geq 1 & \text{if } N \geq 18. \end{cases}$$

Then $N_{\max} = 18$.

Solution 11.4 We have that the likelihood function is given by:

$$L(X_1, ..., X_n, \alpha) = \prod_{i=1}^{n} \exp(\alpha - X_i)\mathbb{1}_{\{X_i \geq \alpha\}},$$
$$= \exp(n\alpha - \sum_{i=1}^{n} X_i)\mathbb{1}_{\{\bigcap_{i=1}^{n} X_i \geq \alpha\}},$$

we note that $f(\alpha) := \exp(n\alpha - \sum_{i=1}^{n} X_i) > 0$ is increasing, so its maximum is attained at the maximum point where $\mathbb{1}_{\{\bigcap_{i=1}^{n}\{X_i \geq \alpha\}\}} \neq 0$. Then the point that maximizes the likelihood is $\alpha = \min_{i=1,..,n}\{X_i\}$.

Solution 11.5 Let μ and σ^2 denote the mean and the variance of X_i, the density of X_i is

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

The 0.95 quantile of X_i is the value $\theta = \theta(\mu, \sigma)$ such that

$$\mathbb{P}(X < \theta) = 0.95 = \mathbb{P}\left(\frac{X-\mu}{\sigma} < \frac{\theta-\mu}{\sigma}\right),$$

thus $\theta_0 = \frac{\theta-\mu}{\sigma}$ which implies that $\theta = \sigma\theta_0 + \mu$.

The logarithm of the product of the X_i's density is

$$l(\mu, \sigma^2) := -\frac{n}{2}\ln(2\pi\sigma^2) - \sum \frac{(X_i-\mu)^2}{2\sigma^2}.$$

We look for μ and σ^2 which maximizes $L(\mu, \sigma^2)$.

$$\frac{\partial L}{\partial \mu} = \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma^2} = 0$$

$$\frac{\partial L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \left(\sum_{i=1}^{n} \frac{(X_i-\mu)^2}{2}\right)\frac{1}{(\sigma^2)^2} = 0.$$

Which give, for any σ^2, $\mu \mapsto L(\mu, \sigma^2)$ is maximized when

$$\mu = \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i,$$

which is the sample average of X_i. And for $\mu = \overline{X}$, $\sigma^2 \mapsto L(\mu, \sigma^2)$ is maximized when

$$\sigma^2 = \sum_{i=1}^{n} \frac{(X_i - \overline{X})^2}{n} = \overline{(X_i - \overline{X})^2},$$

the sample variance.

We substitute the values of μ and σ^2 into the expression of θ and gives the M.L.E. of 0.95-quantile of X:

$$\theta = \theta_0\sqrt{\overline{(X_i - \overline{X})^2}} + \overline{X}.$$