Probability and Statistics

Solution sheet 12

Solution 12.1

$$T = \frac{\sigma^2}{20}\sum_{i=1}^{20}\left(\frac{X_i-\mu}{\sigma}\right)^2.$$

To find the 0.95-quantile of T, remark that

$$\mathbb{P}[T \leq c] = \mathbb{P}\left[\frac{20}{\sigma^2}T \leq \frac{20}{\sigma^2}c\right].$$

Where $\frac{20}{\sigma^2}T \sim \chi^2(20)$. We find in the table of the inverse cumulative distribution function of the Chi-squared distribution that the 0.95-quantile for $\chi^2(20)$ is 31.41. Hence

$$\frac{20}{\sigma^2}c = 31.41 \Rightarrow c = 31.41 \times 0.09/20 = 0.1413.$$

Solution 12.2 Let $(X_i)_{i=1}^n$ be an i.i.d sequence of Bernoulli(p) random variables. Let $X = \sum_{i=1}^{100} X_i$, then $X \sim Bin(100, p)$.

(a) We want to know whether the coin is fair or not, so our hypothesis are

$$H_0 : \text{The coin is fair},$$
$$H_1 : \text{The coin is not fair}.$$

That is to say

$$H_0 : p = p_0 = \frac{1}{2},$$
$$H_1 : p \neq p_0.$$

Then we should use a two-sided test.

Under H_0 we have that $\mathbb{E}_0(X) = 50$ and $Var_0(X) = np_0(1-p_0) = 25$. By central limit theorem, the distribution of $\frac{X-50}{5}$ can be approximated by the standard normal distribution. Let Φ be the c.d.f. of the standard normal distribution. Now, take c_1 and c_2 such that

$$\begin{aligned} 0.01 &\geq \mathbb{P}_0(X \notin (c_1, c_2)) \\ &= 1 - \mathbb{P}_0\left(\frac{c_1-50}{5} \leq \frac{X-50}{5} \leq \frac{c_2-50}{5}\right) \\ &\simeq 1 - \Phi\left(\frac{c_2-50}{5}\right) + \Phi\left(\frac{c_1-50}{5}\right). \end{aligned}$$

We would like to make the rejection zone as large as possible, given that Φ is symmetric, we choose

$$\frac{c_1-50}{5} = -\frac{c_2-50}{5},$$
$$\Rightarrow c_1 = 100 - c_2.$$

Finally

$$\begin{aligned}
&0.01 \geq 1 - \Phi\left(\frac{c_2 - 50}{5}\right) + 1 - \Phi\left(\frac{c_2 - 50}{5}\right), \\
\Rightarrow &\Phi\left(\frac{c_2 - 50}{5}\right) \geq 0.995 \\
\Rightarrow &\frac{c_2 - 50}{5} \geq 2.5758 \\
\Rightarrow &c_2 \geq 62.9,
\end{aligned}$$

then

$$K_{1\%} = [0; 37] \cup [63; 100].$$

Given that $60 \notin K_{1\%}$ we cannot reject H_0 with 0.01 level of significance.

(b) We have to take our hypothesis

$$\begin{aligned}
H_0 &: p = p_0 = \frac{1}{2} \\
H_1 &: p > p_0,
\end{aligned}$$

now we have to find c such that

$$0.01 \geq \mathbb{P}_0(X \geq c) \simeq 1 - \Phi\left(\frac{c - 50}{5}\right).$$

Then we have to choose $c \geq 61.6$, from what we have that $K_{1\%} = [62; 100]$. So we reject H_0 if we have 62 or more heads.

(c) Take $p_0 \in (0, 1)$ and

$$\begin{aligned}
H_0 &: p = p_0, \\
H_1 &: p \neq p_0.
\end{aligned}$$

Note that $\mathbb{E}_0(X) = 100p_0$ and $Var_0(X) = 100p_0(1 - p_0)$. Then, take c_1 and c_2 such that

$$\begin{aligned}
0.01 &\geq \mathbb{P}_0(X \notin (c_1, c_2)) \\
&= 1 - \mathbb{P}_0\left(\frac{c_1 - 100p_0}{10\sqrt{p_0(1 - p_0)}} \leq \frac{X - 100p_0}{10\sqrt{p_0(1 - p_0)}} \leq \frac{c_2 - 100p_0}{10\sqrt{p_0(1 - p_0)}}\right) \\
&\simeq 1 - \Phi\left(\frac{c_2 - 100p_0}{10\sqrt{p_0(1 - p_0)}}\right) + \Phi\left(\frac{c_1 - 100p_0}{10\sqrt{p_0(1 - p_0)}}\right).
\end{aligned}$$

Doing the same that in question a we have that $c_1(p_0) = 100p_0 - 19.6\sqrt{p_0(1 - p_0)}$ and $c_2(p_0) = 100p_0 + 19.6\sqrt{p_0(1 - p_0)}$. Then we are interested in the set

$$\begin{aligned}
&\left\{p : 100p - 19.6\sqrt{p(1 - p)} \leq 60 \leq 100p + 19.6\sqrt{p(1 - p)}\right\} \\
=&\left\{p : (60 - 100p)^2 \leq 19.6^2 p(1 - p)\right\} \\
=&[0.502; 0.691].
\end{aligned}$$

Solution 12.3

(a) The null and alternative hypotheses are

$$H_0 : \mu = \mu_0 (= 1.0085) \quad \text{and} \quad H_A : \mu = \mu_A (= 1.008).$$

We can use the test statistic $T = \bar{X}_n = \frac{\sum_{i=1}^n X_i}{n}$, which has under H_0 distribution $\mathcal{N}(\mu_0, \sigma/\sqrt{n})$. Lastly the rejection rule is that the null hypothesis is rejected if $T \leq c_\alpha$, with c_α determined by

$$P(T \leq c_\alpha) = \int_{-\infty}^{c_\alpha} f_T(t)dt = \alpha (= 0.05).$$

We find $t = \bar{x}_n = \frac{\sum_{i=1}^n x_i}{n} \approx 1.0076$, and we have also

$$P(T \leq c_\alpha) = P\left(Z \leq \sqrt{n}\frac{c_\alpha - \mu_0}{\sigma}\right) = \alpha,$$

where Z has a standard normal distribution under H_0. Hence, using the table, we obtain $\sqrt{n}\frac{c_\alpha - \mu_0}{\sigma} = -1.645$ thus $c_\alpha \approx 1.0078$. Consequently, since $t < c_\alpha$, we reject the null hypothesis H_0.

(b) The power of the test is equal to the probability that the null hypothesis is rejected given that the alternative hypothesis is true:

$$\beta = P_A(T \leq c_\alpha) = P_A\left(Z \leq \sqrt{n}\frac{c_\alpha - \mu_A}{\sigma}\right),$$

where Z has a standard normal distribution under H_A. We have calculated in part **a)** $c_\alpha \approx 1.0078$, thus we get

$$\begin{aligned}\beta &= P_A\left(Z \leq \sqrt{n}\frac{c_\alpha - \mu_A}{\sigma}\right) \approx P_A(Z \leq -0.5) = 1 - P_A(Z \geq -0.5) \\ &= 1 - P_A(Z \leq 0.5) = 1 - 0.6915 = 0.3085\end{aligned}$$

(c) Changing to $\mu_A = 1.007$ leads then to a higher power of the test. A similar calculation provides

$$\beta = P_A\left(Z \leq \sqrt{n}\frac{c_\alpha - \mu_A}{\sigma}\right) \approx P_A(Z \leq 1.98) = 0.9761,$$

which is much better!

Solution 12.4

(a) Under the hypothesis that X has a Poisson distribution, then $\lambda = E[X]$. We use then as an estimator

$$\hat{\lambda} = \frac{\sum_{k=1}^{19} k \cdot N_K}{400} = \frac{\sharp\text{Colonies}}{\sharp\text{Squares}} = 2.44.$$

The expected number of squares with k bacterial colonies is then given by

$$E_k = 400 \cdot \mathbb{P}(X = k) \approx 400 \cdot \frac{\widehat{\lambda}^k e^{-\widehat{\lambda}}}{k!} = \widehat{E}_k.$$

(b) From part **a)**, we get the following table:

k	$\widehat{E}_k$	$\widehat{E}_k - N_k$
0	34.86	-21.14
1	85.07	-18.93
2	103.78	23.78
3	84.41	22.41
4	51.49	9.49
5	25.13	-1.87
6	10.22	1.22
7	3.56	-5.44
8	1.09	-3.91
9	0.29	-2.70
10	0.07	-1.93
19	0	-1

At first glance, the deviations seem fairly large in absolute value, hence the assumption of X having a Poisson distribution with parameter $\lambda = 2.44$ appears inaccurate. A statistical test should thus be performed.

(c) For the last group $\{k \geq 8\}$, we have 11 observations and an expected number of bacterial colonies given by

$$400 \cdot \mathbb{P}(X \geq 8) = 400(1 - \mathbb{P}(X \leq 7)) = 400(1 - \sum_{k=0}^{7} \mathbb{P}(X = k)) \approx 1.47.$$

Hence we get the following values

k	(Obs. − Exp.)^2/Exp.
0	12.82
1	4.21
2	5.45
3	5.95
4	1.75
5	0.14
6	0.15
7	8.31
≥ 8	61.64

Note that the particular way of grouping doesn't influence too much the value of T_N, thus T_N would be large in any case. Lastly, we obtain the test statistic

$$T_N = \sum \frac{(\text{Obs.} - \text{Exp.})^2}{\text{Exp.}} = 100.41.$$

(d) If the value of T_N is too large (i.e. above a critical value c_α), we consider the hypothesis that X has a Poisson distribution with parameter $\lambda = 2.44$ to be inaccurate. The critical value c_α are defined by $\int_{c_\alpha}^{\infty} f_8(x)dx = \alpha$, where $f_8(x)$ is the density of a χ^2 distribution with 8 degrees of freedom. The table gives us then for $\alpha = 1\%$, a critical value of $c_\alpha = 20.090$. Therefore for both levels of confidence $\alpha = 5\%$ and 1%, the test tells us that the hypothesis is inaccurate, and should thus be rejected.

Solution 12.5 Given $n = 25$, the mean of X_i is μ and variance is 1,

$$\overline{X_n} \sim \mathcal{N}(\mu, \sigma^2/n) = \mathcal{N}(\mu, 1/25).$$

Thus $5(\overline{X_n} - \mu) \sim \mathcal{N}(0, 1)$. Let Y be a standard normal distributed random variable and Φ its cumulative distribution function:

$$\begin{aligned}\pi(0.1|\delta) &= \mathbb{P}[\overline{X_n} \le c_1|\mu = 0.1] + \mathbb{P}[\overline{X_n} \ge c_2|\mu = 0.1]\\ &= \mathbb{P}[5\overline{X_n} - 0.5 \le 5c_1 - 0.5|\mu = 0.1] + \mathbb{P}[5\overline{X_n} - 0.5 \ge 5c_2 - 0.5|\mu = 0.1]\\ &= \mathbb{P}[Y \le 5c_1 - 0.5] + \mathbb{P}[Y \ge 5c_2 - 0.5]\\ &= \Phi(5c_1 - 0.5) + (1 - \Phi(5c_2 - 0.5)).\end{aligned}$$

Similarly,

$$\pi(0.2|\delta) = \Phi(5c_1 - 1) + (1 - \Phi(5c_2 - 1)).$$

We have to find c_1 and c_2 such that

$$\Phi(5c_2 - 0.5) - \Phi(5c_1 - 0.5) = 0.93 = \Phi(y + 0.5) - \Phi(x + 0.5) \tag{1}$$

and

$$\Phi(5c_2 - 1) - \Phi(5c_1 - 1) = 0.93 = \Phi(y) - \Phi(x) \tag{2}$$

where we set $x = 5c_1 - 1$ and $y = 5c_2 - 1$.

Taking the difference of (1) and (2), one gets:

$$\Phi(y + 0.5) - \Phi(y) = \Phi(x + 0.5) - \Phi(x).$$

By studying the monotonicity of the function $t \mapsto \Phi(t + 0.5) - \Phi(t)$, any value can have at most two preimages, and the fact that $0.93 = \Phi(y) - \Phi(x)$ implies $x \neq y$. Hence $x + 0.5 = -y$. Substitute it in (2):

$$0.93 = 1 - \Phi(-y) - \Phi(x) = 1 - \Phi(x + 0.5) - \Phi(x)$$

$$0.07 = \Phi(x + 0.5) + \Phi(x).$$

It has a unique solution $x = x_0$ (by monotonicity of $x \mapsto \Phi(x + 0.5) + \Phi(x)$). And $y = -x_0 - 0.5$.

Hence $c_1 = x_0/5 + 1/5$ and $c_2 = -x_0/5 + 1/10$.

Standard normal (cumulative) distribution function.

$$\mathbb{P}(X \leq x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) \mathrm{d}y, \quad \text{for } x \geqslant 0$$

	0	1	2	3	4	5	6	7	8	9
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6408	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
2.2	.98610	.98645	.98679	.98713	.98745	.98778	.98809	.98840	.98870	.98899
2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.99430	.99446	.99461	.99477	.99492	.99506	.99520
2.6	.99534	.99547	.99560	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.99720	.99728	.99736
2.8	.99744	.99752	.99760	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3.0	.998650	.998694	.998736	.998777	.998817	.998856	.998893	.998930	.998965	.998999
3.1	.999032	.999065	.999096	.999126	.999155	.999184	.999211	.999238	.999264	.999289
3.2	.999313	.999336	.999359	.999381	.999402	.999423	.999443	.999462	.999481	.999499
3.3	.999517	.999534	.999550	.999566	.999581	.999596	.999610	.999624	.999638	.999651
3.4	.999663	.999675	.999687	.999698	.999709	.999720	.999730	.999740	.999749	.999758
3.5	.999767	.999776	.999784	.999792	.999800	.999807	.999815	.999822	.999828	.999835
3.6	.999841	.999847	.999853	.999858	.999864	.999869	.999874	.999879	.999883	.999888
3.7	.999892	.999896	.999900	.999904	.999908	.999912	.999915	.999918	.999922	.999925
3.8	.999928	.999931	.999933	.999936	.999938	.999941	.999943	.999946	.999948	.999950
3.9	.999952	.999954	.999956	.999958	.999959	.999961	.999963	.999964	.999966	.999967

The chi-square distribution

$$\Pr(X \leq x) = \int_0^x \frac{1}{\Gamma(r/2)2^{r/2}} \, y^{r/2-1} e^{-y/2} \, \mathrm{d}y$$

	$\Pr(X \leq x)$					
r	0.01	0.025	0.05	0.95	0.975	0.99
1	0.000	0.001	0.004	3.841	5.024	6.635
2	0.020	0.051	0.103	5.991	7.378	9.210
3	0.115	0.216	0.352	7.815	9.348	11.345
4	0.297	0.484	0.711	9.488	11.143	13.277
5	0.554	0.831	1.145	11.070	12.833	15.086
6	0.872	1.237	1.635	12.592	14.449	16.812
7	1.239	1.690	2.167	14.067	16.013	18.475
8	1.646	2.180	2.733	15.507	17.535	20.090
9	2.088	2.700	3.325	16.919	19.023	21.666
10	2.558	3.247	3.940	18.307	20.483	23.209
11	3.053	3.816	4.575	19.675	21.920	24.725
12	3.571	4.404	5.226	21.026	23.337	26.217
13	4.107	5.009	5.892	22.362	24.736	27.688
14	4.660	5.629	6.571	23.685	26.119	29.141
15	5.229	6.262	7.261	24.996	27.488	30.578
16	5.812	6.908	7.962	26.296	28.845	32.000
17	6.408	7.564	8.672	27.587	30.191	33.409
18	7.015	8.231	9.390	28.869	31.526	34.805
19	7.633	8.907	10.117	30.144	32.852	36.191
20	8.260	9.591	10.851	31.410	34.170	37.566
21	8.897	10.283	11.591	32.671	35.479	38.932
22	9.542	10.982	12.338	33.924	36.781	40.289
23	10.196	11.689	13.091	35.172	38.076	41.638
24	10.856	12.401	13.848	36.415	39.364	42.980
25	11.524	13.120	14.611	37.652	40.646	44.314
26	12.198	13.844	15.379	38.885	41.923	45.642
27	12.879	14.573	16.151	40.113	43.195	46.963
28	13.565	15.308	16.928	41.337	44.461	48.278
29	14.256	16.047	17.708	42.557	45.722	49.588
30	14.953	16.791	18.493	43.773	46.979	50.892