Probability and Statistics

Solution sheet 4

Solution 4.1

(a) Let $(A_n)_{n\in\mathbb{N}}$ so that $A_n \in \bigcap_{i\in I} \mathcal{A}_i$ then:

- For all i, $A_1 \in \mathcal{A}_i$, so $A_1^c \in \mathcal{A}_i$. Then $A_1^c \in \bigcap_{i\in I} \mathcal{A}_i$.
- For all i, $A_n \in \mathcal{A}_i$, so $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}_i$. Then $\bigcup_{n\in\mathbb{N}} A_n \in \bigcap_{i\in I} \mathcal{A}_i$.
- For all i, $\Omega \in \mathcal{A}_i$, Then $\Omega \in \bigcap_{i\in I} \mathcal{A}_i$.

Then $\bigcap_{i\in I} \mathcal{A}_i$ is a σ-algebra.

(b) $\Leftarrow$ It is clear that if $\mathcal{A}_1 \subseteq \mathcal{A}_2$, then $\mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A}_2$ and it is a σ-algebra. Idem if $\mathcal{A}_2 \subseteq \mathcal{A}_1$.

$\Rightarrow$ Note first that for $A, B \subseteq \Omega$, we have

$$A\Delta B = (A \cup B) \cap (A \cap B)^c,$$

So if A and B belong to a σ-algebra, their symmetric difference does too.

Now, if $\mathcal{A}_1 \subsetneq \mathcal{A}_2$ and $\mathcal{A}_2 \subsetneq \mathcal{A}_1$, then there exists $A_1 \in \mathcal{A}_1 \backslash \mathcal{A}_2$ and $A_2 \in \mathcal{A}_2 \backslash \mathcal{A}_1$. Note that $A_1 \Delta A_2 \notin \mathcal{A}_i$, because, if $A_1 \Delta A_2 \in \mathcal{A}_i$, then for $i \in \{1, 2\}$

$$A_i \Delta (A_1 \Delta A_2) \in \mathcal{A}_i \quad \text{which implies} \quad A_j \in \mathcal{A}_i \quad j \neq i$$

Then $\mathcal{A}_1 \cup \mathcal{A}_2$ is not a σ-algebra because $A_1 \Delta A_2 \notin \mathcal{A}_1 \cup \mathcal{A}_2$ even though $A_1, A_2 \in \mathcal{A}_1 \cup \mathcal{A}_2$.

(c) (i) $\bigcap_{i\in\mathbb{N}} A_i^c \in \mathcal{A}$. For $n \in \mathbb{N}$, it holds, $\bigcap_{i\in\mathbb{N}} A_i^c \notin \sigma(\{A_k : k \geq n\})$.
This implies $\bigcap_{i\in\mathbb{N}} A_i^c \notin \mathcal{A}_\infty$.

(ii) $\bigcap_{n\in\mathbb{N}} \bigcup_{i\geq n} A_i \in \mathcal{A}$. For $m \in \mathbb{N}$, $\bigcap_{n\in\mathbb{N}} \bigcup_{i\geq n} A_i \in \sigma(\{A_k : k \geq m\})$.
Then $\bigcap_{n\in\mathbb{N}} \bigcup_{i\geq n} A_i \in \mathcal{A}_\infty$.

(iii) $\bigcup_{n\in\mathbb{N}} \bigcap_{i\geq n} A_i^c \in \mathcal{A}$. For $m \in \mathbb{N}$, $\bigcup_{n\in\mathbb{N}} \bigcap_{i\geq n} A_i^c \in \sigma(\{A_k : k \geq m\})$.
Then $\bigcup_{n\in\mathbb{N}} \bigcap_{i\geq n} A_i^c \in \mathcal{A}_\infty$.

(iv) $\bigcup_{j_1<j_2\in\mathbb{N}} A_{j_2} \cap A_{j_1} \cap \bigcap_{\substack{i\in\mathbb{N} \\ j_1\neq i\neq j_2}} A_j^c \in \mathcal{A}$. It holds that $\bigcup_{j_1<j_2\in\mathbb{N}} A_{j_2} \cap A_{j_1} \cap \bigcap_{\substack{i\in\mathbb{N} \\ j_1\neq i\neq j_2}} A_j^c$ does not belong to $\sigma(\{A_k : k \geq 2\})$, so it does not belong to $\mathcal{A}_\infty$.

(v) $\bigcup_{n\in\mathbb{N}} \bigcup_{j_1<j_2<\ldots<j_{2n+1}} \left(\bigcap_{i=1}^{2n+1} A_{j_i} \cap \bigcap_{\substack{i\in\mathbb{N} \\ n\neq j_k}} A_i^c \right) \in \mathcal{A}$. Similarly

$$\bigcup_{n\in\mathbb{N}} \bigcup_{j_1<j_2<\ldots<j_{2n+1}} \left(\bigcap_{i=1}^{2n+1} A_{j_i} \cap \bigcap_{\substack{i\in\mathbb{N} \\ n\neq j_k}} A_i^c \right)$$

does not belong to $\sigma(\{A_k : k \geq 2\})$, so it does not belong to $\mathcal{A}_\infty$.

Solution 4.2

(a) Take $([0,1], \mathcal{B}(0,1), \lambda)$ as a probability space, where $\mathcal{B}[0,1]$ is the Borel σ-algebra on $[0,1]$. Let U be the identity function. U is distributed as an uniform random variable on $(0,1)$ under λ. Define

$$A_n := \left\{ x \in (0,1) : U(x) \in \left[0, \frac{1}{n}\right] \right\}.$$

Then we have that $\mathbb{P}(A_n) = \frac{1}{n}$, so $\sum_{n\in\mathbb{N}} \mathbb{P}(A_n) = \infty$. Additionally $x \in \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k$ if and only if $x = 0$. Therefore, $\mathbb{P}\left(\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k\right) = 0$.

(b) (i) We will use Borel-Cantelli's Lemma. Define $A_n^\alpha := \{U_n < n^{-\alpha}\}$, then:

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n^\alpha) = \sum_{n=1}^{\infty} \frac{1}{n^\alpha} < \infty,$$

and by Borel-Cantelli lemma $\mathbb{P}(\bigcap_{n\in\mathbb{N}} \bigcup_{j\geq n} A_j^\alpha) = 0$. Thus

$$\mathbb{P}\left(\bigcup_{\substack{\alpha>1\\ \alpha\in\mathbb{Q}}} \bigcap_{n\in\mathbb{N}} \bigcup_{j\geq n} A_j^\alpha \right) = 0.$$

Let $\omega \in \Omega$ so that there exists $\alpha(\omega)$ for which $\liminf n^{\alpha(\omega)} U_n(\omega) < \infty$. Then take $1 < \tilde{\alpha}(\omega) < \alpha(\omega)$ with $\tilde{\alpha}(\omega) \in \mathbb{Q}$. We have that $\liminf n^{\tilde{\alpha}(\omega)} U_n(\omega) = 0$. Then for all $n \in \mathbb{N}$ there exists $m(\omega) > n$ so that $m^{\tilde{\alpha}} U_m(\omega) < 1$. Thus, $\omega \in \bigcup_{\substack{\alpha>1\\ \alpha\in\mathbb{Q}}} \bigcap_{n\in\mathbb{N}} \bigcup_{j\geq n} A_j^\alpha$. Finally we have that

$$\{(\exists \alpha > 1) \liminf n^\alpha U_n \in \mathbb{R}\} \subseteq \bigcup_{\substack{\alpha>1\\ \alpha\in\mathbb{Q}}} \bigcap_{n\in\mathbb{N}} \bigcup_{j\geq n} A_j^\alpha$$

This implies

$$\mathbb{P}\left((\exists \alpha > 1) \liminf n^\alpha U_n \in \mathbb{R}\right) = 0.$$

(ii) We use Borel-Cantelli's Lemma. Define $A_n = \{U_n \leq n^{-1}\}$, it is clear that the $(A_n)_{n\in\mathbb{N}}$ are independent. We have $\mathbb{P}(A_n) = \frac{1}{n}$, then $\sum_{n\in\mathbb{N}} \mathbb{P}(A_n) = \infty$. By Borel-Cantelli

$$\mathbb{P}\left(\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k \right) = 1 > 0.$$

Additionally, if $\omega \in \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k$, for all $n \in \mathbb{N}$ there exists $k_n(\omega) > n$ such that $k_n(\omega) U_{k_n(\omega)} \leq 1$. Thus, $0 \leq \liminf nU_n \leq 1$. To conclude,

$$\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k \subseteq \{\liminf nU_n \in \mathbb{R}\} \quad \text{and} \quad \mathbb{P}\left(\liminf nU_n \in \mathbb{R}\right) = 1 > 0.$$

Solution 4.3

(a) The projection $X_n(\omega) : (\omega_1, \omega_2, ..., \omega_n, ...) \mapsto \omega_n$ is measurable, so, for $k \in \mathbb{N}$, the random variable $\sum_{n=1}^{k} \frac{X_n}{3^n}$ is measurable (as a sum of random variables). Given that

$$\sum_{n=1}^{k} \frac{X_n}{3^n} \overset{k\to\infty}{\to} X, \text{ point-wise},$$

X is measurable.

(b) The cumulative function is continuous in x if and only if $\mathbb{P}[\{X(\omega) = x\}] = 0$. This is clear in this case.

(c) Define $K_1 = (\frac{1}{3}, \frac{2}{3})$. It holds that $K_1 \cap X(\{0,2\}^{\mathbb{N}}) = \emptyset$. Indeed, if $X_1(\omega) = 0$, then we have

$$X(\omega) = \sum_{n=2}^{\infty} \frac{X_n(\omega)}{3^n} \leq \frac{2}{9} \frac{1}{1-\frac{1}{3}} = \frac{1}{3},$$

and if $X_1(\omega) = 2$ then

$$X(\omega) = \frac{2}{3} + \sum_{n=2}^{\infty} \frac{X_n(\omega)}{3^n} \geq \frac{2}{3}.$$

Define $L_1 = [0,1] \backslash K_1$, $a_1^{(1)} = 0$ and $a_2^{(1)} = 2/3$, then we know that $L_1 = \bigcup_{i=1}^{2^1} [a_i^{(1)}, a_i^{(1)} + (\frac{1}{3})^1]$. Given $L_{n-1} = \bigcup_{i=1}^{2^{n-1}} [a_i^{(n-1)}, a_i^{(n-1)} + \left(\frac{1}{3}\right)^{n-1}]$, $a_i^{(n-1)}$ for $i \in \{1, ..., 2^{n-1}\}$, we define K_n, $\left(a_i^{(n)}\right)_{i \in \{1,\ldots,2^n\}}$ as follows

$$\begin{aligned} K_n &= \bigcup_{i=1}^{2^{n-1}} \left(a_i^{(n-1)} + \left(\frac{1}{3}\right)^n, a_i^{(n-1)} + 2\left(\frac{1}{3}\right)^n \right) \\ a_{2i-1}^{(n)} &= a_i^{(n-1)} \\ a_{2i}^{(n)} &= a_i^{(n-1)} + 2\left(\frac{1}{3}\right)^n \end{aligned}$$

Then $L_n = L_{n-1} \backslash K_n$.

We claim the following

(i) K_n is well defined.

(ii) $K_n \cap X(\{0,2\}^{\mathbb{N}}) = \emptyset$, so $X(\{0,2\}^{\mathbb{N}}) \subseteq L_n$.

(iii) $\lambda(L_n) = \left(\frac{2}{3}\right)^n$.

Indeed,

(i) Note that L_1 is well defined and is of the desired form. If L_{n-1} is of the wanted form, then K_n divides every interval in 3 and takes away the middle part, so the number of intervals will be $2 \cdot 2^{n-1} = 2^n$, all of them having length $\frac{1}{3^{n-1}} \cdot \frac{1}{3} = \frac{1}{3^n}$. So L_n has the desired form.

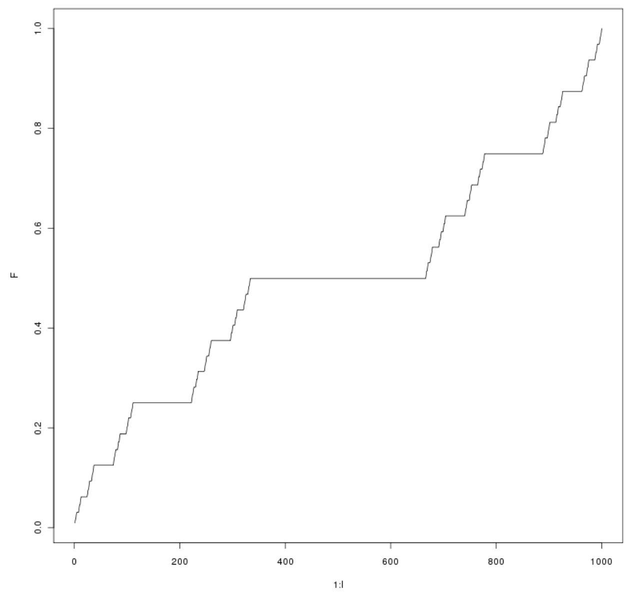
(ii) Realize that all $a_i^{(n)}$ have the form

$$a_i^n := \sum_{i=1}^{n} \frac{\epsilon_i}{3^n},$$

where $\epsilon_i \in \{0,2\}$. As for K_1, we have the equation $\bigcup_{n\in\mathbb{N}} K_n \cap X(\{0,2\}^{\mathbb{N}}) = \emptyset$ Then

$$X(\{0,2\}^{\mathbb{N}}) \subseteq \left(\bigcup_{n=1}^{N} K_n \right)^c = L_N$$

(iii) L_n has 2^n disjoint intervals, each one with measure $\frac{1}{3^n}$ so $\lambda(L_n) = (\frac{2}{3})^n$.



Thanks to this

$$1 \geq \lambda(\bigcup_{n\in\mathbb{N}} K_n) \geq 1 - \lambda(L_n) \to 1.$$

As $K_n = \bigcup_k I_k^{(n)}$ we have the conclusion.
To finish here is a graph of the cumulative function, called Devil's Staircase: