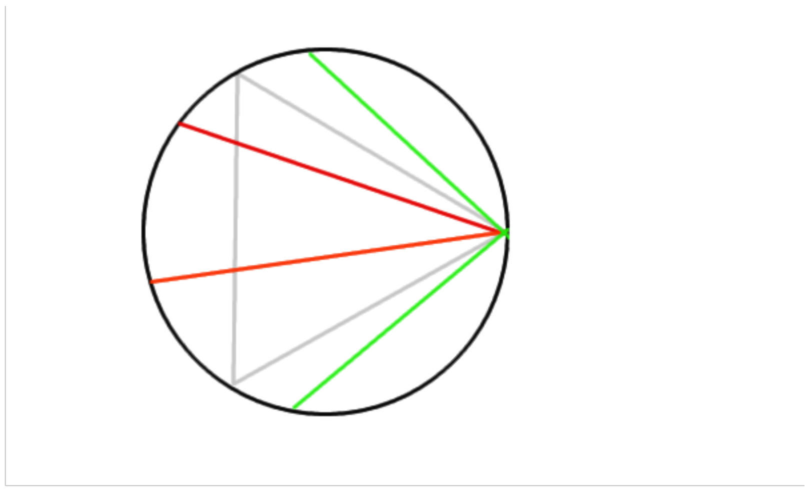
Probability and Statistics

Solution sheet 5

Solution 5.1

(a) Remember that $x \mod 1 = x - \lfloor x \rfloor$. Note that the chord is longer than a side of the triangle if $(V-U) \mod 1 \in \left(\frac{1}{3}, \frac{2}{3}\right)$.



$$
\begin{aligned}
\mathbb{P}\left[(V-U) \mod 1 \in \left(\frac{1}{3}, \frac{2}{3}\right)\right] &= \iint_{(0,1)^2} \mathbb{1}_{(y-x) \mod 1 \in \left(\frac{1}{3}, \frac{2}{3}\right)} dy dx \\
&= \int_0^1 \left(\int_0^1 \mathbb{1}_{y \in \left(\frac{1}{3}+x, \frac{2}{3}+x\right) \mod 1} dy \right) dx \\
&= \int_0^1 \frac{1}{3} dy = \frac{1}{3}.
\end{aligned}
$$

(b) Let r be the point chosen in the radius, it is a uniform random variable over $[0,1]$. The length of the chord will be given by $2\sqrt{(1-r^2)}$. Then

$$
\mathbb{P}\left(2\sqrt{(1-r^2)} \geq \sqrt{3}\right) = \mathbb{P}\left(1 - r^2 \geq \frac{3}{4}\right) = \mathbb{P}\left(r^2 \leq \frac{1}{4}\right) = \frac{1}{2}.
$$

(c) Let (x,y) be the point chosen in the circle, it is a uniform random variable over $B(0,1)$. The length of the chord will be given by $2\sqrt{1-(x^2+y^2)}$. Thus,

$$
\begin{aligned}
\mathbb{P}\left(2\sqrt{(1-(x^2+y^2))} \geq \sqrt{3}\right) &= \mathbb{P}\left(1 - (x^2+y^2) \geq \frac{3}{4}\right) \\
&= \mathbb{P}\left((x^2+y^2) \leq \frac{1}{4}\right) \\
&= \iint_{B(0,\frac{1}{2})} \frac{1}{\pi} dx dy = \frac{1}{4}.
\end{aligned}
$$

(d) This is not a contradiction. What this shows us is that there is not a formal way to "pick a chord uniformly at random" in this circle, so we have to define the probability measure we are interested in before asking the question about the probability of an event.

Solution 5.2 Let us suppose, first, that $\mu_1 = \mu_2 = 0$. We just have to use the convolution formula:

$$\begin{aligned}
f_{X_1+X_2}(x) &= \frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} \exp(-\frac{y^2}{2\sigma_1^2})\exp(-\frac{(x-y)^2}{2\sigma_2^2})dy \\
&= \frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} \exp\left(-\frac{\sigma_2^2 y^2 + \sigma_1^2(x-y)^2}{2\sigma_1^2\sigma_2^2}\right)dy \\
&= \frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} \exp\left(-\frac{\sigma_2^2 y^2 + \sigma_1^2 x^2 + \sigma_1^2 y^2 - 2\sigma_1^2 xy}{2\sigma_1^2\sigma_2^2}\right)dy \\
&= \frac{\exp(-\frac{x^2}{2\sigma_2^2})}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} \exp\left(-(\sigma_1^2+\sigma_2^2)\frac{y^2 - \frac{2\sigma_1^2 xy}{(\sigma_1^2+\sigma_2^2)}}{2\sigma_1^2\sigma_2^2}\right)dy \\
&= \frac{\exp(-\frac{x^2}{2\sigma_2^2} + \frac{\sigma_1^2 x^2}{2\sigma_2^2(\sigma_1^2+\sigma_2^2)})}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} \exp\left(-\frac{\left(y - \frac{\sigma_1^2 x}{(\sigma_1^2+\sigma_2^2)}\right)^2}{\frac{2\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}}\right)dy \\
&= \frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_2^2}}\exp(-\frac{x^2}{2(\sigma_1^2+\sigma_2^2)}),
\end{aligned}$$

that is the distribution function of a normal random variable with parameter $N(0, \sigma_1^2 + \sigma_2^2)$.

For the general case note, that $X_i - \mu_i$ is distributed as $N(0, \sigma_i^2)$. So $(X_1 - \mu_1) + (X_2 - \mu_2) \sim N(0, \sigma_1^2 + \sigma_2^2)$, then $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Solution 5.3

(a) We have that the cumulative distribution function of Y for $y \geq 0$ is given by:

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y}) = 2F_X(\sqrt{y}) - 1.$$

Then, taking the derivative we have:

$$f_Y(y) = f_x(\sqrt{y})y^{-\frac{1}{2}}\mathbb{1}_{y\geq 0} = ce^{-\frac{y}{2}}y^{-\frac{1}{2}}\mathbb{1}_{y\geq 0}.$$

(b) By the convolution formula we have that:

$$\begin{aligned}
f_{Y_1+Y_2}(x) &= \int_0^x f_Y(x-y)f_Y(y)dy\mathbb{1}_{x\geq 0} \\
&= c_1^2\int_0^x (x-y)^{-\frac{1}{2}}e^{-\frac{x-y}{2}}y^{-\frac{1}{2}}e^{-\frac{y}{2}}dy\mathbb{1}_{x\geq 0} \\
&= c_1^2 e^{-\frac{x}{2}}\int_0^x (x-y)^{-\frac{1}{2}}y^{-\frac{1}{2}}dy\mathbb{1}_{x\geq 0} \\
&= c_1^2\left(\int_0^1 x(x-ux)^{-\frac{1}{2}}(ux)^{-\frac{1}{2}}du\right)e^{-\frac{x}{2}}\mathbb{1}_{x\geq 0} \\
&= \left(c_1^2\int_0^1 (1-u)^{-\frac{1}{2}}u^{-\frac{1}{2}}du\right)e^{-\frac{x}{2}}\mathbb{1}_{x\geq 0}.
\end{aligned}$$

This is the distribution of an exponential random variable.

(c) By the previous questions, the base case is true. Now let us prove the inductive step. Suppose that the proposition is true for $n-1$, then

$$\begin{aligned}
f_{\sum_{i=1}^n Y_i}(x) &= \int_0^x f_Y(x-y) f_{\sum_{i=1}^{n-1} Y_i}(y) dy \mathbb{1}_{x\geq 0} \\
&= c_1 c_{n-1} \int_0^x (x-y)^{-\frac{1}{2}} e^{-\frac{x-y}{2}} y^{\frac{n-1}{2}-1} e^{-\frac{y}{2}} dy \mathbb{1}_{x\geq 0} \\
&= c_1 c_{n-1} e^{-\frac{x}{2}} \int_0^1 (x-xu)^{-\frac{1}{2}} (xu)^{\frac{n-1}{2}-1} x du \mathbb{1}_{x\geq 0} \\
&= \left(c_1 c_{n-1} \int_0^1 (1-u)^{-\frac{1}{2}} (u)^{-\frac{n-1}{2}-1} du \right) e^{-\frac{x}{2}} x^{\frac{n}{2}-1} \mathbb{1}_{x\geq 0}.
\end{aligned}$$

Solution 5.4

(a) To find the density we differentiate the cumulative distribution function

$$F(t) := \mathbb{P}(X \leq t) = 1 - \mathbb{P}(X \geq t) = 1 - e^{-\lambda t}.$$

Then its density is

$$f(t) := F'(t) = \lambda e^{-\lambda t}.$$

We can calculate its mean as

$$\begin{aligned}
\mathbb{E}(X) &= \int_0^\infty t\lambda e^{-\lambda t} dt \\
&= -te^{-\lambda t} \mid_0^\infty + \int_0^\infty e^{-\lambda t} dt \\
&= \frac{1}{\lambda}.
\end{aligned}$$

Its second moment is

$$\begin{aligned}
\mathbb{E}\left(X^2\right) &= \int_0^\infty \lambda t^2 e^{-\lambda t} dt \\
&= -t^2 e^{-\lambda t} \mid_0^\infty + 2\int_0^\infty te^{-\lambda t} dt \\
&= \frac{2}{\lambda^2}.
\end{aligned}$$

Finally,

$$Var(X) = \mathbb{E}\left(X^2\right) - \mathbb{E}(X)^2 = \frac{1}{\lambda^2}.$$

(b) We have to compute

$$\begin{aligned}
\mathbb{P}(\min\{X_1, X_2\} > t) &= \mathbb{P}(X_1 > t, X_2 > t) \\
&= \mathbb{P}(X_1 > t)\mathbb{P}(X_2 > t) \\
&= e^{-\lambda_1 t} e^{-\lambda_2 t} \\
&= e^{-(\lambda_1+\lambda_2)t}.
\end{aligned}$$

This is the definition of $\min\{X_1, X_2\} \sim \mathcal{E}(\lambda_1 + \lambda_2)$.

(c) It holds that,

$$\mathbb{P}\left(Y \geq t+h \mid Y \geq h\right) = \frac{\mathbb{P}(Y \geq t+h)}{\mathbb{P}(Y \geq h)} = e^{-\lambda t} = \mathbb{P}(Y \geq t).$$

(d) We have

$$\begin{aligned} G(t+h) &= \mathbb{P}(Y \geq t+h) \\ &= \frac{\mathbb{P}(Y \geq t+h)}{\mathbb{P}(Y \geq h)} \mathbb{P}(Y \geq h) \\ &= \mathbb{P}(Y \geq t+h \mid Y \geq h)\mathbb{P}(Y \geq h) \\ &= G(t)G(h). \end{aligned}$$

(e) First we will prove by induction that for all $n \in \mathbb{N}$ and $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ we have that $G(\sum_{i=1}^n a_i) = \prod_{i=1}^n G(a_i)$. It holds when $n = 1$. Then, assuming the statement is true for some $n \geq 1$

$$G\left(\sum_{i=1}^{n+1} a_i\right) = G(a_{n+1})G\left(\sum_{i=1}^{n} a_n\right) = \prod_{i=1}^{n+1} G(a_i),$$

where we first used the memoryless property and then the induction hypothesis. Take $m, n \in \mathbb{N}$, we have that

$$\begin{aligned} G(1)^m = G\left(\sum_{i=1}^{m} 1\right) = G\left(\sum_{i=1}^{n} \frac{m}{n}\right) = G\left(\frac{m}{n}\right)^n \\ \Rightarrow G(1)^{\frac{m}{n}} = G\left(\frac{m}{n}\right) \end{aligned}$$

(f) Finally, take $t \in \mathbb{R}^+$ and $(t_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}} \subseteq \mathbb{Q}$ so that $t_n \nearrow t$ and $s_n \searrow t$. The monotonicity of $G(t)$ yields

$$\begin{aligned} G(t_n) \leq G(t) \leq G(s_n) \\ G(1)^{t_n} \leq G(t) \leq G(1)^{s_n} \end{aligned}$$

which gives $G(t) = G(1)^t$. Finally we have that $\mathbb{P}(Y \geq t) = G(1)^t = e^{-\ln\left(\frac{1}{G(1)}\right)t}$, then $Y \sim \mathcal{E}\left(\ln\left(\frac{1}{G(1)}\right)\right)$.

Solution 5.5 We compute first the cumulative distribution function of λ given $X = 1$. By hypothesis,

$$\mathbb{P}(X = 1 \,|\, \Lambda) = \mathbb{E}(\mathbb{1}_{\{X=1\}} \,|\, \Lambda) = e^{-\Lambda}\Lambda.$$

Hence

$$\begin{aligned} \mathbb{P}(X = 1) = \mathbb{E}(\mathbb{1}_{\{X=1\}}) = \mathbb{E}[\mathbb{E}(\mathbb{1}_{\{X=1\}} \,|\, \Lambda)] &= \int_0^\infty e^{-\lambda}\lambda f(\lambda) d\lambda \\ &= \int_0^\infty 2e^{-\lambda}\lambda e^{-2\lambda} d\lambda \\ &= \int_0^\infty 2\lambda e^{-3\lambda} d\lambda = 2/9. \end{aligned}$$

By definition of the conditional expectation, and that for any $x > 0$, $\mathbb{1}_{\{\Lambda \leq \lambda\}}$ is $\sigma(\Lambda)$-measurable, hence

$$\mathbb{P}(X = 1, \Lambda \leq \lambda) = \mathbb{E}[\mathbb{1}_{\{X=1\}}\mathbb{1}_{\{\Lambda\leq\lambda\}}] = \mathbb{E}[\mathbb{E}(\mathbb{1}_{\{X=1\}} \,|\, \Lambda)\mathbb{1}_{\{\Lambda\leq\lambda\}}] = \mathbb{E}[e^{-\Lambda}\Lambda\mathbb{1}_{\{\Lambda\leq\lambda\}}],$$

and the cumulative distribution function of λ given $X = 1$ is

$$\mathbb{P}(\Lambda \leq \lambda \,|\, X = 1) = \frac{\mathbb{P}(X = 1, \Lambda \leq \lambda)}{\mathbb{P}(X = 1)} = \frac{\int_0^\lambda 2se^{-3s} ds}{P(X = 1)}.$$

Differentiating the cumulative distribution function with respect to λ, we obtain the probability distribution function

$$f(\lambda \,|\, X = 1) = 9\lambda e^{-3\lambda}.$$