Probability and Statistics

Solution sheet 7

Solution 7.1

(a) The distribution function of an exponentially distributed random variable X is given by

$$F_X(x) = \mathbb{P}(X \leq x) = 1 - e^{-\lambda x}.$$

For an α-quantile q_α one has

$$\mathbb{P}(X \leq q_\alpha) \geq \alpha \text{ and } \mathbb{P}(X \geq q_\alpha) \geq 1 - \alpha.$$

Since X has a density (so $\mathbb{P}(X = x) = 0$ for every $x \in \mathbb{R}$),

$$1 - e^{-\lambda q_\alpha} = \mathbb{P}(X \leq q_\alpha) = \alpha.$$

That is

$$q_\alpha = -\ln(1 - \alpha)/\lambda.$$

(b) The distribution function is

$$F_U(u) = \mathbb{P}(U \leq u) = \begin{cases} 0, & u < 1 \\ k/N, & u \in [k, k+1),\, k \in \{1, \cdots N-1\} \\ 1, & u \geq N \end{cases}.$$

Then, if $\alpha = k/N$ for some $k \in \{1, \cdots, N-1\}$, the quantile can be chosen in $[k, k+1)$. If $\alpha \in (k/N, (k+1)/N)$, then the quantile is uniquely determined: $q_\alpha = k + 1$.

Solution 7.2

(a) We have to prove that $\int_{\mathbb{R}^2} f_{X,Y}(x, y)dxdy = 1$.

$$\begin{aligned}\int_1^\infty \int_0^\infty e^{-x^2 y} dydx &= \int_1^\infty \left(-\frac{e^{-x^2 y}}{x^2} \right)\Bigg|_0^\infty dx \\ &= \int_1^\infty \frac{1}{x^2} dx = 1.\end{aligned}$$

(b) By definition, $f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x, y)dy$, then

$$f_X(x) = \int_0^\infty e^{-x^2 y} dy \mathbb{1}_{x \geq 1} = \frac{1}{x^2} \mathbb{1}_{x \geq 1}.$$

(c) We compute

$$\begin{aligned}\mathbb{P}\left(Y \leq \frac{1}{X^2}\right) &= \int_1^\infty \int_0^{1/x^2} e^{-x^2 y} dxdy \\ &= \int_1^\infty \frac{1}{x^2} - e^{-1} \frac{1}{x^2} = 1 - e^{-1}.\end{aligned}$$

Solution 7.3

(a) $X \sim \mathcal{E}(\lambda)$ so it has density

$$f_X(x) = \mathbb{1}_{\{x \geq 0\}} \lambda e^{-\lambda x},$$

Y as well,

$$f_Y(y) = \mathbb{1}_{\{y \geq 0\}} \lambda e^{-\lambda y}.$$

The joint density of $(X, Z) := (X, X+Y)$ is

$$\begin{aligned} f_{X,Z}(x,z) &= f_{X,Y}(x, z-x) \\ \text{(independence of } X \text{ and } Y) &= f_X(x) f_Y(z-x) \\ &= \mathbb{1}_{\{x \geq 0\}} \lambda e^{-\lambda x} \mathbb{1}_{\{z-x \geq 0\}} \lambda e^{-\lambda(z-x)} \\ &= \mathbb{1}_{\{0 \leq x \leq z\}} \lambda^2 e^{-\lambda z} \end{aligned}$$

(b) The event B has probability:

$$\begin{aligned} \mathbb{P}(B) &= \int_a^\infty \int_0^a f_{X,Z}(x,z) dx dz \\ &= \int_a^\infty \int_0^a \mathbb{1}_{\{0 \leq x \leq z\}} \lambda^2 e^{-\lambda z} dx dz \\ &= a\lambda^2 \int_a^\infty e^{-\lambda z} dz \\ &= a\lambda e^{-\lambda a}. \end{aligned}$$

Note that it is also the probability that a Poisson($a\lambda$) random variable equals to 1 (it is also the probability of a Poisson process with intensity λ having exactly one point on the interval $[0, a]$). Similarly, for $0 \leq b \leq a$,

$$\begin{aligned} \mathbb{P}(X \leq b \,|\, B) &= \frac{\mathbb{P}(X \leq b, Z \geq a)}{\mathbb{P}(B)} \\ &= \frac{\int_a^\infty \int_0^b f_{X,Z}(x,z) dx dz}{a\lambda e^{-\lambda a}} \\ &= \frac{b}{a}. \end{aligned}$$

(c) The density is obtained by taking the derivative of $\mathbb{P}(X \leq b \,|\, B)$ with respect to b, which gives

$$f_{X|B}(x) = \mathbb{1}_{\{0 \leq x \leq a\}} \frac{1}{a}.$$

We conclude that the conditional law of X given B is the uniform distribution on $[0, a]$.

Solution 7.4

(a) For $\lambda = 0$ the property is trivial. If $\lambda > 0$, using Markov (Tchebyshev) inequality we have that

$$\begin{aligned} \mathbb{P}(X \geq t) &= \mathbb{P}(\exp(\lambda X) \geq \exp(\lambda t)) \\ &\leq \frac{1}{\exp(\lambda t)} \mathbb{E}\left(\exp(\lambda X)\right). \end{aligned}$$

(b) Given that $\exp(\lambda\cdot)$ is a convex function, thanks to Jensen inequality we have that:

$$\mathbb{E}(\exp(\lambda X)) \geq \exp(\mathbb{E}(\lambda X))$$

and so

$$\phi(\lambda) = \ln\left(\mathbb{E}(\exp(\lambda X))\right) \geq \lambda \mathbb{E}\left(X\right).$$

(c) Using part a) we have that for all $\lambda > 0$

$$\begin{aligned}
\mathbb{P}(X \geq \lambda) &\leq \exp(-\lambda t)\mathbb{E}\left(e^{\lambda X}\right) = \exp(\phi_X(\lambda) - \lambda t) \\
\mathbb{P}(X \geq \lambda) &\leq \exp(-\sup_{\lambda \geq 0}\{\lambda t - \phi_X(\lambda)\}),
\end{aligned}$$

where we have just taken the infimum of the exponential term over $\lambda \geq 0$.

(d) Let us compute

$$\begin{aligned}
\mathbb{E}\left(e^{\lambda X}\right) &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp(\lambda x)\exp(\frac{-x^2}{2\sigma^2})dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp(-\frac{x^2 - 2\lambda\sigma^2}{2\sigma^2})dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp(-\frac{(x-\lambda\sigma^2)^2}{2\sigma^2})\exp(\frac{\lambda^2\sigma^2}{2})dx \\
&= \exp(\frac{\lambda^2\sigma^2}{2}),
\end{aligned}$$

then $\phi(\lambda) = \frac{1}{2}\lambda^2\sigma^2$.

(e) We use Fubini's theorem to compute

$$\begin{aligned}
\mathbb{E}\left(X\right) &= \mathbb{E}\left(\int_0^{\infty} \mathbb{1}_{\{t \leq X\}} dt\right) \\
&= \int_0^{\infty} \mathbb{E}\left(\mathbb{1}_{\{X \geq t\}}\right) dt \\
&= \int_0^{\infty} \mathbb{P}(X \geq t) dt.
\end{aligned}$$

(f) First, we need to calculate

$$\sup_{\lambda > 0}\left\{\lambda t - \frac{1}{2}\lambda^2\sigma^2\right\},$$

this is just a quadratic function in λ, so it attains its maximum at $\lambda_{\max} = t/\sigma^2 > 0$. Then its supremum is $t^2/2\sigma^2$. With the result of question (c), we obtain

$$\begin{aligned}
\mathbb{P}(Y \geq t) &\leq \exp(-\sup_{\lambda \geq 0}\{\lambda t - \phi_Y(\lambda)\}) \\
&\leq \exp(-\sup_{\lambda \geq 0}\{\lambda t - \phi_X(\lambda)\}) \\
&= \exp\left(-\frac{t^2}{2\sigma^2}\right).
\end{aligned}$$

Thus, using question (e) we have

$$\begin{aligned}\mathbb{E}\left(Y^2\right) &= \int_0^\infty P(Y^2 \geq t)dt = \int_0^\infty P(Y \geq \sqrt{t})dt + \int_0^\infty P(-Y \geq \sqrt{t})dt \\ &\leq 2\int_0^\infty \exp\left(-\frac{t}{2\sigma^2}\right)dt \\ &= 4\sigma^2.\end{aligned}$$

Where we have used also

$$\phi_{-Y}(\lambda) = \phi_Y(-\lambda) \leq \phi_X(-\lambda) = \phi_{-X}(\lambda) = \phi_X(\lambda),$$

hence the previous step can also be applied to $\mathbb{P}(-Y \geq \sqrt{t})$.

Solution 7.5

(a)
- $\varphi_X(0) = \mathbb{E}[1] = 1$.
- $\varphi_X(t) = \int e^{itx}\mu(dx) \leq \int |e^{itx}|\mu(dx) = 1$, since μ is a probability measure on $\mathbb{R}$.
- This follows from the classical dominated convergence theorem and that e^{itx} is bounded.
- $\varphi_{aX+b}(t) = \mathbb{E}[e^{it(aX+b)}] = e^{itb}\mathbb{E}[e^{itaX}] = e^{itb}\varphi_X(at)$.

(b) We can prove this by induction. The case $k = 0$ holds by definition of φ_X. Assume that for $k < n$ the proposition is true, and consider the quantity

$$\frac{\varphi_X^{(k)}(t+h) - \varphi_X^{(k)}(t)}{h} = i^k\mathbb{E}[X^k e^{itX} X \frac{e^{ihX}-1}{hX}].$$

Since $|\frac{e^{ihX}-1}{hX}| \leq 1$ for any X and goes to i as $h \to 0$, using the fact that X^{k+1} is integrable, we obtain that the above expression tends to $i^{k+1}\mathbb{E}[X^{k+1}e^{itX}]$, by the dominated convergence theorem.

(c) Let X be a standard Normal random variable (i. e. $X \sim \mathcal{N}(0,1)$),

$$\begin{aligned}\varphi_X(t) &= \int_{-\infty}^{+\infty} e^{itx}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\ &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-(x^2-2itx)/2}dx \\ &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-(x-it)^2/2}e^{-t^2/2}dx \\ &= e^{-t^2/2}.\end{aligned}$$

If $Y \sim \mathcal{N}(\mu, \sigma^2)$, then $(Y-\mu)/\sigma \sim \mathcal{N}(0,1)$. Thus

$$\varphi_Y(t) = e^{it\mu}\varphi_X(\sigma t) = e^{it\mu - \sigma^2 t^2/2}.$$

(d) As X and Y are independent, for every $f, g : \mathbb{R} \to \mathbb{R}$, $f(X)$ and $g(Y)$ are also independent random variables. Thus

$$\varphi_{X+Y}(t) = \mathbb{E}[e^{it(X+Y)}] = \mathbb{E}[e^{itX}e^{itY}] = \mathbb{E}[e^{itX}]\mathbb{E}[e^{itY}] = \varphi_X(t)\varphi_Y(t).$$

Solution 7.6

(a) We have that

$$\mathbb{E}\left(S_n^4\right) = \frac{1}{n^4}\sum_{i,j,k,l=1}^{n} \mathbb{E}\left(X_i X_j X_k X_l\right),$$

note that if $i \notin \{j,k,l\}$

$$\mathbb{E}\left(X_i X_j X_k X_l\right) = \mathbb{E}\left(X_i\right)\mathbb{E}\left(X_j X_k X_l\right) = 0.$$

Thus,

$$\begin{aligned}\mathbb{E}\left(X_i X_j X_k X_l\right) &= \frac{1}{n^4}\sum_{i,j,k,l=1}^{n} \mathbb{1}_{\{i=j=k=l\}}\mathbb{E}\left(X_1^4\right) \\ &\quad + \frac{1}{n^4}\sum_{i,j,k,l=1}^{n}(\mathbb{1}_{\{i=j\neq k=l\}} + \mathbb{1}_{\{i=k\neq j=l\}} + \mathbb{1}_{\{i=l\neq k=j\}})\mathbb{E}\left(X_1^2\right)^2 \\ &= \frac{1}{n^3}\mathbb{E}\left(X_1^4\right) + \frac{6(n-1)}{n^3}\mathbb{E}\left(X_1^2\right)^2.\end{aligned}$$

We have by Cauchy-Schwarz inequality that

$$\mathbb{E}\left(X_1^2\right) = \mathbb{E}\left(X_1^2 * 1\right) \leq \sqrt{\mathbb{E}\left(X_1^4\right)}\sqrt{\mathbb{E}\left(1^2\right)} < \infty$$

(b) We will use the Markov inequality, i.e.,

$$\begin{aligned}\mathbb{P}\left(|S_n| > a\right) &= \mathbb{P}\left(\frac{(S_n)^4}{a^4} > 1\right) \\ &= \mathbb{E}\left[\mathbb{1}_{\{\frac{(S_n)^4}{a^4}\geq 1\}}\right] \\ &\leq \mathbb{E}\left[\frac{(S_n)^4}{a^4}\mathbb{1}_{\{\frac{(S_n)^4}{a^4}\geq 1\}}\right] \\ &\leq \mathbb{E}\left[\frac{(S_n)^4}{a^4}\right] \\ &= \frac{1}{a^4}\left(\frac{1}{n^3}\mathbb{E}\left(X^4\right) + \frac{6(n-1)}{n^3}\mathbb{E}\left(X_1^2\right)^2\right) \\ &\leq \frac{6}{a^4}\frac{1}{n^2}\mathbb{E}\left[X_1^4\right],\end{aligned}$$

where in the last inequality we have used that $\mathbb{E}\left[X_1^2\right]^2 \leq \mathbb{E}\left[X_1^4\right]$.

(c) Take $A_n^m = \{\omega : |S_n(\omega)| > \frac{1}{m}\}$. We have that

$$\sum_{n\in\mathbb{N}} \mathbb{P}(A_n^m) \leq \sum_{n\in\mathbb{N}} 6m^4\frac{1}{n^2}\mathbb{E}\left(X^4\right) < \infty.$$

By Borel-Cantelli $\mathbb{P}(\bigcap_{n\in\mathbb{N}}\bigcup_{k\geq n} A_k) = 0$, so

$$\mathbb{P}\left(\bigcup_{m\in\mathbb{N}}\bigcap_{n\in\mathbb{N}}\bigcup_{k\geq n} A_k\right) = 0$$

$$\mathbb{P}\left(\bigcap_{m\in\mathbb{N}}\bigcup_{n\in\mathbb{N}}\bigcap_{k\geq n} A_k^c\right) = 1.$$

If $\omega \in \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{n\geq k} A_k^c$ then for all $m \in \mathbb{N}$ there exists $n(\omega)$ so that for all $k \geq n(\omega)$ $|S_n(\omega)| < \frac{1}{m}$. Thus, $\lim_{n\to\infty} |S_n(\omega)| = 0$. This implies that

$$\mathbb{P}\left(\lim S_n(\omega) = 0\right) = 1.$$

(d) Define $\tilde{X}_n = X_n - \mathbb{E}\left(X_n\right)$. We have that $\tilde{X}_n$ satisfies all the hypothesis for (c), then

$$\mathbb{P}\left(\lim \tilde{X}_n = 0\right) = 1$$
$$\Rightarrow \mathbb{P}\left(\lim_{n\to\infty} X_n = \mathbb{E}\left[X_n\right]\right) = 1.$$