Probability and Statistics

Solution sheet 8

Solution 8.1

(a) We compute:

$$\begin{aligned}\mathbb{P}(Z \geq t) &= \mathbb{P}(\{X \geq t, Y > 0\} \cup \{X \leq -t, Y \leq 0\}) \\ &= \frac{1}{2}\mathbb{P}(X \geq t) + \frac{1}{2}\mathbb{P}(X \leq -t) \\ &= \mathbb{P}(X \geq t).\end{aligned}$$

Therefore, Z has the same law as X, thus $Z \sim \mathcal{N}(0,1)$.

(b) Using the definition of covariance

$$\begin{aligned}\mathrm{Cov}(X,Z) &= \mathbb{E}\left(XZ\right) - \mathbb{E}(X)\mathbb{E}(Z) \\ &= \mathbb{E}\left(X^2 \mathbb{1}_{\{y\geq 0\}}\right) + \mathbb{E}\left(-X^2 \mathbb{1}_{\{y<0\}}\right) = 0.\end{aligned}$$

(c) We have that

$$\mathbb{P}(X+Z=0) = \mathbb{P}(Y<0) + \mathbb{P}(Y \geq 0, 2X = 0) = \frac{1}{2}.$$

(d) The vector (X,Z) is not a multivariate normal, because the sum of the two coordinates is not normally distributed.

Solution 8.2

(a) As X is a Gaussian vector, $Y := \sum_{i=1}^n \alpha_i X_i$ is also Gaussian. $\mathbb{E}(Y) = 0$ and

$$\mathrm{Var}(Y) = \mathrm{Cov}\left(\sum_{i=1}^n \alpha_i X_i, \sum_{j=1}^n \alpha_j X_j\right).$$

The covariance operator is bilinear, hence

$$\mathrm{Var}(Y) = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j K_X(i,j) = \alpha K_X\,{}^t\alpha,$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$.

(b) K_X is symmetric, and as the variance is always non-negative, K_X is positive.

(c) First, $Z := (X_1, X_2)$ is a Gaussian vector. Let denote Σ its variance matrix.

$$\Sigma = \begin{pmatrix}\mathrm{Var}(X_1) & 0 \\ 0 & \mathrm{Var}(X_2)\end{pmatrix}$$

The characteristic function of Z is

$$\begin{aligned}
\varphi_Z(a,b) &= \mathbb{E}[\mathrm{e}^{i(aX_1+bX_2)}] \\
&= \frac{1}{2\pi\sqrt{|\Sigma|}} \iint \mathrm{e}^{i(ax+by)} \exp\left(-\frac{1}{2}(x,y)\begin{pmatrix}\mathrm{Var}(X_1) & 0 \\ 0 & \mathrm{Var}(X_2)\end{pmatrix}^{-1}\begin{pmatrix}x \\ y\end{pmatrix}\right) dxdy \\
&= \exp\left(-(a,b)\begin{pmatrix}\mathrm{Var}(X_1) & 0 \\ 0 & \mathrm{Var}(X_2)\end{pmatrix}\binom{a}{b}/2\right) \\
&= \exp\left(-(a^2\mathrm{Var}(X_1)+b^2\mathrm{Var}(X_2))/2\right) \\
&= \exp\left(-a^2\mathrm{Var}(X_1)/2\right)\exp\left(-b^2\mathrm{Var}(X_2)/2\right).
\end{aligned}$$

You can compare this result with Exercise 7.5.

The above expression is also the characteristic function of the joint law of two independent centered random gaussian variables, with variance respectively $\mathrm{Var}(X_1)$ and $\mathrm{Var}(X_2)$. Since the characteristic function determines the joint law, X_1 and X_2 are independent. If we omit the hypothesis that (X_1, X_2) is a Gaussian vector, then 0 covariance does not imply the independence, as the example given in the previous exercise shows.

Solution 8.3 We apply the formula

$$\mathrm{Cov}(X_1,X_2) = \mathbb{E}(X_1X_2) - \mathbb{E}(X_1)\mathbb{E}(X_2).$$

Since the marginal density for X_1 is

$$f_1(x_1) = \int_0^1 4x_1x_2dx_2 = 2x_1,$$

$$\mathbb{E}(X_1) = \int_0^1 x_1f_1(x_1)dx_1 = 2/3 = \mathbb{E}(X_2).$$

We have also

$$\mathbb{E}(X_1X_2) = \int_0^1\int_0^1 x_1x_24x_1x_2dx_1dx_2 = 4/9.$$

Thus $Cov(X_1,X_2) = 0$. Actually, one can see from $f_1(x_1)f_2(x_2) = f(x_1,x_2)$ that X_1 and X_2 are independent.

We want to find the joint p.d.f. of $(Y_1,Y_2) = (r_1(X_1,X_2), r_2(X_1,X_2))$, where $r_1(x_1,x_2) = x_1/x_2$ and $r_2(x_1,x_2) = x_1x_2$. We have that

$$x_1 = \sqrt{r_1r_2} \text{ and } x_2 = \sqrt{r_2/r_1}.$$

$$Jac(r_1,r_2) = det\begin{pmatrix}\partial x_1/\partial r_1 & \partial x_2/\partial r_1 \\ \partial x_1/\partial r_2 & \partial x_2/\partial r_2\end{pmatrix} = \frac{1}{4}det\begin{pmatrix}\sqrt{r_2/r_1} & -\sqrt{r_2/r_1^3} \\ \sqrt{r_1/r_2} & \sqrt{1/r_1r_2}\end{pmatrix} = \frac{1}{2r_1}.$$

By the transformation formula of the density,

$$\begin{aligned}
f_{Y_1,Y_2}(r_1,r_2) &= f_{X_1,X_2}(\sqrt{r_1r_2},\sqrt{r_2/r_1})Jac(r_1,r_2) \\
&= 4r_2/2r_1\mathbb{1}_{\{0<\sqrt{r_1r_2},\sqrt{r_2/r_1}<1\}} \\
&= \mathbb{1}_{\{0<\sqrt{r_1r_2},\sqrt{r_2/r_1}<1\}}2r_2/r_1.
\end{aligned}$$

Solution 8.4

(a) For $k = 1, 2, \cdots$

$$\begin{aligned}\mathbb{E}[X^k] &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^k x^{\alpha-1} e^{-\beta x} dx \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+k-1} e^{-\beta x} dx \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \beta^{-\alpha-k} y^{\alpha+k-1} e^{-y} dy \\ &= \frac{\Gamma(\alpha+k)}{\beta^k \Gamma(\alpha)},\end{aligned}$$

where we used the change of variable $y = \beta x$. Then $\mathbb{E}(X) = \alpha/\beta$. Furthermore, for $\alpha > 0$,

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha).$$

We then deduce

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \alpha/\beta^2.$$

(b) By definition of moment generating function,

$$\begin{aligned}\psi_X(t) &= \mathbb{E}(e^{tX}) \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{tx} x^{\alpha-1} e^{-\beta x} dx \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\beta-t)x} dx \\ &= \left(\frac{\beta}{\beta - t}\right)^\alpha\end{aligned}$$

The above computation is only valid for $t < \beta$.

(c) If ψ_i denotes the moment generating function of X_i, then it follows from the last question that for $i = 1, \cdots, k$,

$$\psi_i(t) = \left(\frac{\beta}{\beta - t}\right)^{\alpha_i}.$$

The m.g.f. ψ of $X_1 + \cdots + X_k$ is, by independence,

$$\psi(t) = \prod_{i=1}^k \psi_i(t) = \left(\frac{\beta}{\beta - t}\right)^{\alpha_1+\cdots+\alpha_k} \quad \text{for } t < \beta.$$

It coincides with the m.g.f. of a Gamma random variable with parameter $(\alpha_1 + \cdots + \alpha_k, \beta)$, on an open interval around 0, thus the sum $X_1 + \cdots + X_k$ has the Gamma distribution.

Solution 8.5

(a) The joint probability distribution function of $X_1, \cdots, X_n, Z$ is

$$\begin{aligned}f(x_1, \cdots, x_n, z) &= \prod_{i=1}^n g_1(x_i|z) f_2(z) \\ &= 2z^n \exp(-z[2 + x_1 + \cdots + x_n]),\end{aligned}$$

if $z, x_1, \cdots, x_n > 0$ and 0 otherwise.

(b) The marginal joint distribution of $X_1, \cdots, X_n$ is obtained by integrating z out of the joint probability distribution function above.

$$\int_0^\infty f(x_1, \cdots, x_n, z)dz = \frac{2\Gamma(n+1)}{(2+x_1+\cdots+x_n)^{n+1}} = \frac{2(n!)}{(2+x_1+\cdots+x_n)^{n+1}},$$

for all $x_i > 0$ and 0 otherwise.

(c) We set $y = 2 + \sum_{i=1}^n x_i$, for $z > 0$

$$\begin{aligned} g_2(z|x_1, \cdots, x_n) &= f(x_1, \cdots, x_n, z)\frac{y^{n+1}}{2(n!)} \\ &= \frac{z^n \exp(-zy)y^{n+1}}{n!} \\ &= \frac{y^{n+1}}{\Gamma(n+1)} z^{n+1-1}e^{-yz}, \end{aligned}$$

we recognize the conditional distribution of Z given $X_1 = x_1, \cdots X_n = x_n$ is Gamma distribution with parameter $\alpha = n+1$, $\beta = y$.

(d) The conditional expected value of Z given $X_1 = x_1, \cdots X_n = x_n$ is the expected value of the Gamma distribution with parameter $\alpha = n+1$, $\beta = y$, which by the previous exercise, equals to

$$\mathbb{E}(Z|X_1 = x_1, \cdots X_n = x_n) = \frac{n+1}{2+\sum_{i=1}^n x_i}.$$

Solution 8.6 If we define $S_n := \sum_{i=1}^n X_i \sim Poi(n)$, we have that:

$$e^{-n}\sum_{k=0}^n \frac{n^k}{k!} = \mathbb{P}(S_n \le n) = \mathbb{P}(S_n \le n\mathbb{E}(X_1)) = \mathbb{P}\left(\frac{1}{\sqrt{n}}(S_n - n\mathbb{E}(X_1)) \le 0\right) \to \frac{1}{2}.$$