Probability and Statistics

Solution sheet 9

Solution 9.1

(a) By definition of a lognormal distribution, $N := \log(Z_1) \sim \mathcal{N}(\mu, \sigma^2)$. Thus

$$\mathbb{E}(Z_1) = \mathbb{E}(e^N) = \varphi_N(1) = e^{\mu+\sigma^2/2},$$

where we have used the moment generating function of N:

$$\varphi_N(t) = \mathbb{E}(e^{tN}) = e^{t\mu+t^2\sigma^2/2}.$$

We get:

$$\mathbb{E}(S_1) = S_0 e^{\mu+\sigma^2/2}.$$

(b) Let $X := (\log(Z_1) - \mu)/\sigma \sim \mathcal{N}(0,1)$,

$$S_1 = S_0 e^{\mu+\sigma X} \Rightarrow \frac{1}{S_1} = \frac{e^{-\mu-\sigma X}}{S_0}.$$

Hence $1/S_1$ has the distribution of $1/S_0$ times a lognormal random variable with parameter $-\mu$ and σ^2.

(c) By question a)

$$\mathbb{E}(1/S_1) = e^{-\mu+\sigma^2/2}/S_0.$$

Note that if we use a random variable to modelize the rate of change from A to B, than its inverse will be the rate of change from B to A. The product of expected value is always larger than 1 if the variance is non-zero (Jensen's inequality). Here $\mathbb{E}(S_1)\mathbb{E}(1/S_1) = e^{\sigma^2}$.

(d) The k-th moment of a lognormal distribution is easy to compute: For $k = 1, 2, \cdots$,

$$\mathbb{E}(S_1^k) = S_0^k \mathbb{E}(e^{k\mu+k\sigma X}) = (S_0 e^{\mu})^k e^{k^2\sigma^2/2}.$$

Solution 9.2

(a) Since Z and V are independent, the p.d.f. of the couple (Z, V) is

$$f_{Z,V}(z,v) = f_Z(z) f_V(v),$$

where $f_Z(z) = e^{-z^2/2}/\sqrt{2\pi}$ is the p.d.f. of standard normal distribution. By the transformation formula, one obtains the joint p.d.f. of (T, V):

$$f_{T,V}(t,v) = f_Z\left(t\sqrt{\frac{v}{n}}\right) f_V(v) \sqrt{\frac{v}{n}}.$$

(b) The conditional probability given $V = v$ is

$$f_{T|v}(t) = \frac{f_{T,V}(t,v)}{f_V(v)} = f_Z\left(t\sqrt{\frac{v}{n}}\right)\sqrt{\frac{v}{n}} = \exp\left(-\frac{t^2}{2n/v}\right)\frac{\sqrt{v/n}}{\sqrt{2\pi}},$$

which is the p.d.f. of a centered normal distribution with variance n/v. One can guess it by replace directly V by v in the expression of T (which can be proven when Z and V are independent).

(c) The c_n can be computed as the constant making f_V a p.d.f. (with integral 1):

$$1/c_n = \int_0^\infty x^{n/2-1} \exp(-x/2) dx = 2^{n/2}\Gamma(n/2).$$

(d) The p.d.f. of T is obtained by integrating $f_{T,V}$:

$$\begin{aligned} f_T(t) &= \int_0^\infty f_{T,V}(t,v)dv \\ &= \frac{c_n}{\sqrt{2\pi n}} \int_0^\infty \exp\left(-v\frac{t^2}{2n}\right) v^{n/2-1} e^{-v/2} \sqrt{v} dv \\ &= \frac{c_n}{\sqrt{2\pi n}} \int_0^\infty \exp\left(-v\left(\frac{t^2}{2n} + \frac{1}{2}\right)\right) v^{(n+1)/2-1} dv \\ &= \frac{c_n}{\sqrt{2\pi n}} \frac{\Gamma((n+1)/2)}{\left(\frac{t^2}{2n} + \frac{1}{2}\right)^{(n+1)/2}} \\ &= \left(\frac{t^2}{n} + 1\right)^{-(n+1)/2} \frac{\Gamma((n+1)/2)}{\Gamma(n/2)\sqrt{\pi n}} 2^{-n/2-1/2+(n+1)/2} \\ &= \left(\frac{t^2}{n} + 1\right)^{-(n+1)/2} \frac{\Gamma((n+1)/2)}{\Gamma(n/2)\sqrt{\pi n}}. \end{aligned}$$

Solution 9.3

(a) Set $f = \mathbb{1}_{[-3,1]}$, then

$$A = \mathbb{E}[f(U)] = \mathbb{P}(U \in [-3,1]) = \mathbb{P}[f(U) = 1].$$

(b) nS_n follows the binomial law with parameter (n, A). Hence

$$\mathbb{P}(nS_n = k) = \binom{n}{k} A^k (1-A)^{n-k},$$

or equivalently

$$\mathbb{P}(S_n - A = k/n - A) = \binom{n}{k} A^k (1-A)^{n-k}.$$

(c) $\mathbb{E}(S_n) = A$ and

$$\begin{aligned} \mathrm{Var}(S_n) &= n\mathrm{Var}(f(U_i)/n) \\ &= (A - A^2)/n. \end{aligned}$$

(d) By Tchebychev inequality

$$\mathbb{P}[|S_n - A| \geq x] = \mathbb{P}[|S_n - A|^2 \geq x^2] \leq \frac{\mathrm{Var}(S_n)}{x^2} \leq \frac{1}{nx^2}.$$

The convergence is valid for all $x > 0$, which means that S_n converges in probability to A. To numerically approximate the value A, one can sample independently a family $(U_i)_{i=1..n}$ having the standard normal distribution, and counts the number of points in the interval $[-3,1]$ then divide by n. While the n becomes larger we get a better approximation of A.

(e) We can apply the weak Law of large number.

Solution 9.4 Let X follows Chi-squared distribution $\chi^2(m)$ with m degrees of freedom which is also a Gamma distribution with parameter $(m/2, 1/2)$. Then $\sqrt{X}$ follows $\chi(m)$, has the expected value:

$$\begin{aligned}
\mathbb{E}[\sqrt{X}] &= \frac{(1/2)^{m/2}}{\Gamma(m/2)} \int_0^\infty \sqrt{x} x^{m/2-1} e^{-x/2} dx \\
&= \frac{(1/2)^{m/2}}{\Gamma(m/2)} \int_0^\infty y^{(m+1)/2-1} e^{-y} (1/2)^{-(m+1)/2} dy \\
&= \sqrt{2} \frac{\Gamma((m+1)/2)}{\Gamma(m/2)}.
\end{aligned}$$