Exercise Sheet 1

1. Show that any map $f : S^1 \longrightarrow S^1$ such that $\deg(f) \neq 1$ has a fixed point.

2. Let G be a topological group and take its identity element e as basepoint.

a) Define a pointwise product of loops α and β by $(\alpha\beta)(t) = \alpha(t)\beta(t)$. Prove that it is equivalent to the composition of paths.

b) Deduce that $\pi_1(G, e)$ is abelian.

3. Let $\mathcal{C}$ be a category with a terminal object 1 and all products of objects.

a) Give the definition of a group object on $\mathcal{C}$. Explain how group objects on $\mathcal{C}$ form a category $\underline{\mathrm{Gp}}(\mathcal{C})$.

b) Decide what a group object in $\mathcal{C}$ is when $\mathcal{C}$ is the category:

- of sets;
- of topological spaces;
- of open subsets of a fixed topological space X.

c) Prove that a group object in the category of groups is an abelian group.

d) Let $\mathcal{D}$ be another category like $\mathcal{C}$ and $F : \mathcal{C} \longrightarrow \mathcal{D}$ a functor preserving products and terminal objects. Show that $\mathcal{F}$ induces a functor $\underline{\mathrm{Gp}}(\mathcal{C}) \longrightarrow \underline{\mathrm{Gp}}(\mathcal{D})$.

e) Deduce the same result as in the previous exercise.

4. Find the fundamental group of the torus with two handles:

Please turn over!

5. Find the fundamental group of the Klein bottle.

6. Let $X = \{(p,q) : p \neq -q\} \subseteq S^n \times S^n$ and consider the map $f : S^n \longrightarrow X$ sending $p \mapsto (p,p)$. Prove that it is a homotopy equivalence. [*Hint:* Define a map $X \longrightarrow S^n$ which you could not extend to the whole $S^n \times S^n$]

7. Let $\mathcal{C}$ be a category.

a) Use common sense to complete the following definition: *A* coequalizer *of two morphisms $f, g : X \longrightarrow Y$ in $\mathcal{C}$ is an object C together with a morphism $c : Y \longrightarrow C$ such that...*

b) Define again the coequalizer of f as an initial object of a suitable category.

c) Explain how a cokernel (in an additive category) is a coequalizer.

d) Show: if $\mathcal{C}$ has all coequalizers and arbitrary coproducts, then $\mathcal{C}$ is *cocomplete*, that is, it has all colimits.

e) Deduce formally, in terms of opposite categories, that a category which has all products and equalizers is complete.