Exercise Sheet 11

We will need the following excision theorem (see Theorem 4.23 in Hatcher's book):

Theorem. Let X be a CW complex decomposed as the union of subcomplexes A and B with non-empty connected intersection C. If (A, C) is m-connected and (B, C) is n-connected for some integers $m, n \geq 0$, then the map $\pi_1(A, C) \longrightarrow \pi_1(X, B)$ induced by inclusion is an isomorphism for $i < m + n$ and a surjection for $i = m + n$.

1. Let (X, A) be a CW pair.

a) Consider the space $X' = X \cup CA$ obtained by attaching a cone CA along $A \subseteq X$. Prove that the quotient map $X' \longrightarrow X'/CA = X/A$ is a homotopy equivalence.

Suppose moreover that (X, A) is r-connected and A is s-connected for some $r, s \geq 0$.

b) Show that (CA, A) is $(s + 1)$-connected.

c) Show that the map $\pi_i(X, A) \longrightarrow \pi_i(X/A)$ induced by the quotient $X \longrightarrow X/A$ is an isomorphism for $i \leq r + s$ and a surjection for $i = r + s + 1$. [*Hint:* Excision]

2. Let I be a finite set and $n \geq 2$ an integer. Consider the wedge sum of spheres $X = \bigvee_{i \in I} S^n$. Prove that the inclusions $S^n \hookrightarrow X$ induce an isomorphism $\mathbb{Z}^I \xrightarrow{\sim} \pi_n(X, x_0)$. Generalise this for possibly infinite I. [*Hint:* If you do not know how to proceed, look at Example 4.26 on Hatcher's book]

3. Consider that map $\mathbb{P}^\infty\mathbb{R} \longrightarrow \mathbb{P}^\infty\mathbb{C} = K(\mathbb{Z}, 2)$ induced by the inclusion. What does it induce on homology groups, cohomology groups and homotopy groups?

4. Show that the group structure on S^1 coming from multiplication in $\mathbb{C}$ induces a group structure on $[X, S^1]$, which makes the bijection $\Phi : [X, S^1] \xrightarrow{\sim} H^1(X; \mathbb{Z})$ a group isomorphism.

5. Let G, H be abelian groups and consider CW complexes $K(G, n)$ and $K(H, n)$.

a) What is the canonical map $[K(G, n), K(H, n)] \longrightarrow \mathrm{Hom}(G, H)$?

b) Prove that this map is a bijection.

Please turn over!

6. Let X be a topological space. A *principal G-bundle on X* is a covering $p : E \longrightarrow X$ together with an action of G on E inducing a simply transitive action of G on $p^{-1}(x_0)$ for each $x_0 \in X$.

a) Explain how principal G-bundles form a category. In particular, explain what an isomorphism of principal G-bundles is.

b) Check that the quotient map $p_G : EG \longrightarrow BG$ is a G-bundle.

c) Given a map $f : X \longrightarrow BG$, how do we construct the pullback $f^*p : f^*EG \longrightarrow X$ of p?

d) Prove that this induces an isomorphism

$$[X, BG] \xrightarrow{\sim} \{\text{principal } G\text{-bundles}\}/\sim .$$

7. Let X be a $K(G, 1)$ space and denote by X_n its n-th skeleton component.

a) Prove that $\pi_1(X_1)$ is a free group.

b) Prove that $\pi_2(X_2)$ is a free abelian group group.

c) (*) What can you say about $\pi_n(X_n)$?

Show that $\pi_n(X^n)$ is free abelian for $n \geq 2$.