Exercise Sheet 2

1. Let G be a Hausdorff, connected, locally path connected topological group with identity element e and $p : H \longrightarrow G$ a connected covering. Fix $f \in p^{-1}(e)$.

a) Prove that H has a unique continuous product $H \times H \longrightarrow H$ with identity element f such that p is a homorphism.

b) Show that H is a topological group under this product, and it is abelian if G is.

c) What happens if H is not connected?

d) Prove that $\ker(p)$ is a discrete normal subgroup of H.

e) The map

$$\begin{aligned} \vartheta : \ker(p) &\longrightarrow \mathrm{Aut}_G(H) \\ k &\longmapsto (h \mapsto kh) \end{aligned}$$

is an isomorphism of groups.

f) Deduce that, denoting by $\tilde{G}$ the universal covering of G with its unique groups structure as in **a)**, we have a short exact sequence of groups

$$1 \longrightarrow \pi_1(G, e)^{\mathrm{op}} \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1. \tag{1}$$

2. Let H be a connected locally path connected topological group and $K \trianglelefteq H$ a normal discrete subgroup. Show that $K \trianglelefteq Z(H)$.

3. Let $f : X \longrightarrow Y$ be a map of connected, locally path connected Hausdorff topological spaces. We say that f is a *local homeomorphism* if every point $x \in X$ is an open neighborhood that maps homeomorphically onto an open subset of Y.

a) Give an example of a surjective local homeomorphism that is not a covering.

b) Prove: if X is compact and $f : X \longrightarrow Y$ a surjective local homeomorphism, then f is a covering with finite fibers.