Exercise Sheet 6

1. Give a proof of any of the two following lemmas. Here Ci denotes the homotopy cofiber of a map i, and Fp the homotopy fiber of a map p.

Lemma 1. If $i : A \longrightarrow X$ is a cofibration, then the quotient map

$$\psi : Ci \longrightarrow Ci/CA \cong X/A$$

is a based homotopy equivalence.

Lemma 2. If $p : E \longrightarrow B$ is a fibration, then the inclusion

$$\varphi : p^{-1}(*) \longrightarrow Fp$$

sending $e \mapsto (e, c_*)$ is a based homotopy equivalence.

2. Prove that for $i \geq 2$ one has $\pi_1(\mathbb{P}^i\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ and $\pi_n(\mathbb{P}^i\mathbb{R}) \cong \pi_n(S^i)$ for $n > 1$.

3. Show that for all spaces X and Y and all $n \in \mathbb{Z}_{\geq 0}$ one has

$$\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y).$$

4. Let X be a path connected topological space and $x_0, x_1 \in X$ and let $n \in \mathbb{Z}_{\geq 0}$.

a) Explain how each path γ from x_0 to x_1 induces an isomorphism

$$\Phi_\gamma : \pi_n(X, x_1) \cong \pi_n(X, x_0).$$

b) Show that Φ_γ is invariant under homotopy of γ.

c) Deduce that in the case $x_0 = x_1$ we obtain an action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$.

d) Can you describe this action as the action of the fundamental group on the fiber of a suitable fibration?

e) Show that there is a bijection

$$\{\text{maps } S^n \longrightarrow X\}/\text{homotopy} \xrightarrow{\sim} \pi_n(X,x_0)/\pi_1(X,x_0).$$

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