Exercise Sheet 7

1. Let X be a locally path-connected topological space and $x_0 \in X$. Prove that the Hurewicz map

$$\Phi_1 : \pi_1(X, x_0) \longrightarrow H_1(X, \mathbb{Z})$$

induces an isomorphism

$$\overline{\Phi}_1 : \pi_1(X, x_0)^{\mathrm{ab}} \xrightarrow{\sim} H_1(X, \mathbb{Z}).$$

[See also Exercise Sheet 6, Exercise 1 from Algebraic Topology I]

2. **a)** Let $n > 2$ be an integer. Check that

$$\mathrm{SO}(n-1) \longrightarrow \mathrm{SO}(n) \longrightarrow S^{n-1}$$

is a locally trivial fibration.

b) Compute $\pi_1(\mathrm{GL}(r))$ for each positive integer r.

3. Let r, n be positive integers.

a) Explain in detail how to define the map

$$\{\text{vector bundles on } S^n \text{ of rank } r\}/\cong \; \longrightarrow \pi_{n-1}(\mathrm{GL}(r)).$$

b) Check that it is a bijection of sets.

[If you do not feel familiar with vector bundles, you may want to have a look at Bott, Tu, *Differential Forms in Algebraic Topology*, pages 53-61 and/or at Hirsch, *Differential Topology*, chapter 4]