Exercise Sheet 8

1. Let G be a group.

a) Construct a topological space X with $x_0 \in X$ such that $\pi_1(X, x_0) \cong G$.

b) Suppose that G is finitely presented. Prove that X in the previous part can be taken to be a finite CW complex.

c) Let X be any finite CW complex and $x_0 \in X$. Show that $\pi_1(X, x_0)$ is finitely presented.

2. Let X be a CW complex. You may assume the cellular approximation theorem.

a) Show that the map $\pi_n(X_j) \longrightarrow \pi_n(X)$ is surjective for each $j \geq n$.

b) (*) Show that it is an isomorphism for $j > n$.

3. **a)** Describe explicitly a CW-structure on $\mathbb{P}^1$, $\mathbb{P}^2$ and $\mathbb{P}^3$. Show that $\mathbb{P}^{n+1}$ can be obtained out of $\mathbb{P}^n$ by attaching a $(n+1)$-cell for all $n \geq 1$.

b) How do we define the infinite projective space $\mathbb{P}^\infty\mathbb{R}$?

c) Compute $\pi_n(\mathbb{P}^\infty\mathbb{R})$ for $n \geq 0$.

4. Let $f : X \longrightarrow Y$ be a cellular map between two connected CW complexes. Prove that f factors as a composition $X \longrightarrow Z_n \longrightarrow Y$ where the first map induces isomorphisms on π_i for $i \leq n$ and the second map induces isomorphisms on π_i for $i \geq n+1$.

5. (Yoneda Lemma) Let $\mathcal{C}$ be a category and X an object in $\mathcal{C}$. We denote by h_X the contravariant functor $\mathrm{Hom}(-, X) : \mathcal{C} \longrightarrow \mathrm{Sets}$.

a) Let $F : \mathcal{C} \longrightarrow \mathrm{Sets}$ be a contravariant functor. Prove that the map

$$\begin{aligned}\mathrm{Hom}_{\mathrm{functors}}(h_X, F) &\longrightarrow F(X)\\ \eta &\longmapsto \eta_X(\mathrm{id}_X).\end{aligned}$$

is a bijection.

b) Let X' be another object of $\mathcal{C}$. What happens for $F = h_{X'}$?

Please turn over!

c) We say that a contravariant functor $F : \mathcal{C} \longrightarrow \text{Sets}$ is represented by X if $F \cong h_X$. Deduce from the previous points: if two representable functors are isomorphic, then the corresponding representing objects are isomorphic.

d) Which of the following contravariant functors are representable?

- $\mathcal{P} : \text{Sets} \longrightarrow \text{Sets}$ sending X to its power set $\mathcal{P}(X)$.
- $\text{Open} : \text{Top} \longrightarrow \text{Sets}$ sending each topological space to the set of its open subsets.
- $F : \mathcal{C} \longrightarrow \text{Sets}$, where $\mathcal{C}$ is the category of sets with morphisms given by the injective maps, sending X to the set of partial orders on X.