Exercise Sheet 9

1. (*Compression criterion*) Let X be a topological space, $A \subseteq X$ a closed subset and $x_0 \in A$. Denote by J^n the closure of $\partial I^n \setminus I^n$. Prove that a map

$$f : (I^n, \partial I^n, J^n) \longrightarrow (X, A, x_0)$$

represents the trivial element in $\pi_n(X, A, x_0)$ if and only if it is homotopic relatively to ∂I^n to a map with image in A.

2. Suppose that a sum $f +' g$ of maps $f, g : (I^n, \partial I^n) \longrightarrow (X, x_0)$ is defined using a coordinate of I^n other that the first coordinate as in the usual sum $f + g$. Verify that the formula

$$(f + g) +' (h + k) = (f +' h) + (g +' k)$$

holds and deduce that $f +' k \sim_{\text{htp}} f + k$ so that the two sums agree on $\pi_n(X, x_0)$ and also that $g +' h \sim_{\text{htp}} h + g$ so that the addition is abelian.

State and prove a relative version.

3. For a pair (X, A) of path-connected spaces, show that $\pi_1(X, a, x_0)$ can be identified in a natural way with the set of cosets αH of the subgroup $H \subseteq \pi_1(X, x_0)$ represented by loops in A at x_0.

4. Show that the sequence $\pi_1(X, x_0) \longrightarrow \pi_1(X, A, x_0) \xrightarrow{\partial} \pi_0(A, x_0) \longrightarrow \pi_0(X, x_0)$ is exact.

5. Embed S^{n-1} as equator of S^n and take a point $x_0 \in S^{n-1}$. Compute $\pi_k(S^n, S^{n-1}, x_0)$ for $k \leq n$.

6. **a)** Find an example of spaces $A \subseteq X$ such that $\pi_2(X, A)$ is a non-abelian group.

b) (**) Can you find such an example in which $\pi_1(A)$ is abelian?

7. Let (X, A) be a CW pair. Prove that $A \hookrightarrow X$ is a cofibration.