Exercise 1 by Linda Xiao

2017年3月1日 17:13

1. Show that any map $f : S^1 \longrightarrow S^1$ such that $\deg(f) \neq 1$ has a fixed point.

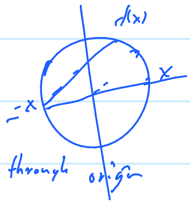
sol: In general, we have

$f: S^n \to S^n$ has no fixed point $\Rightarrow \deg(f) = (-1)^{n+1}$

We restrict the proof to the case $n=1$

s.t. $\pi_1(S^1) \cong H_1(S^1) \cong \mathbb{Z}$

$f(x) \neq x \quad \forall x$



the segment $\overline{-x, f(x)}$ doesn't go through origin

$$\Rightarrow \quad h(x,t) = \frac{(1-t)f(x) - tx}{|(1-t)f(x) - tx|}$$

$h: S^1 \times [0,1] \to S^1$ defines a homotopy from $f(x)$ to antipodal map g

if $f \simeq_{htp} g$ ~ as we already know

$f_* = g_*$

$\Rightarrow \deg(f) = \deg(g) = (-1)^{1+1} = 1$

2. Let G be a topological group and take its identity element e as basepoint.

a) Define a pointwise product of loops α and β by $(\alpha\beta)(t) = \alpha(t)\beta(t)$. Prove that it is equivalent to the composition of paths.

b) Deduce that $\pi_1(G, e)$ is abelian.

Sol: denote the composition of paths by $\alpha * \beta$

$$\alpha * \beta(t) = \begin{cases} \beta(2t) & 0 \leq t \leq \frac{1}{2} \\ \alpha(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

$\alpha \cdot \beta \simeq_{rp} \alpha * \beta$

Want,

$h(t,0) = \alpha(t)\,\beta(t)$

$h(t,1) = \alpha * \beta(t)$

$$h(t,s) = \begin{cases} \beta((1-s+2t)s)\ \alpha((1-s)t)\ \beta((1-s)t) & 0 \le t \le \frac{1}{2}s \\ \alpha((2t-s)s)\ \alpha((1-s)t)\ \beta((1-s)t) & \frac{1}{2}s \le t \le 1 \end{cases}$$

$h(t,0) = \beta(0)\ \alpha(t)\,\beta(t)$

$$h(t,1) = \begin{cases} \beta(2t) & 0 \le t \le \frac{1}{2} \\ \alpha(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

(2)

$[\alpha], [\beta] \in \pi_1(G, e)$

$[\alpha] \cdot [\beta] = [\alpha * \beta]$

want:

$$[\beta] \cdot [\alpha] = [\beta * \alpha] = [\alpha * \beta]$$

$$[\beta * \alpha] = [\beta\alpha], \quad [\alpha * \beta] = [\alpha \cdot \beta]$$

find a homotopy from $\beta\alpha$ to $\alpha\beta$

$\alpha \cdot \beta(t) = \alpha(t)\,\beta(t)$

$H(t,s) = \alpha((1-s)t)\,\beta(t)\,\alpha(st)$

done

$H(t,0) = \alpha(t)\,\beta(t)$

$H(t,1) = \beta(t)\,\alpha(t)$

$\Rightarrow$ $[\alpha \cdot \beta] = [\beta \cdot \alpha]$

$[\alpha * \beta] = [\beta * \alpha]$

$\Rightarrow$ $[\alpha] \cdot [\beta] = [\beta]\,[\alpha]$

For any topological group, $\pi_1(G, e)$ is Abelian

3. Let $\mathcal{C}$ be a category with a terminal object 1 and all products of objects.

a) Give the definition of a group object on $\mathcal{C}$. Explain how group objects on $\mathcal{C}$ form a category $\underline{\mathrm{Gp}}(\mathcal{C})$.

b) Decide what a group object in $\mathcal{C}$ is when $\mathcal{C}$ is the category:

- of sets;
- of topological spaces;
- of open subsets of a fixed topological space X.

c) Prove that a group object in the category of groups is an abelian group.

d) Let $\mathcal{D}$ be another category like $\mathcal{C}$ and $F : \mathcal{C} \longrightarrow \mathcal{D}$ a functor preserving products and terminal objects. Show that $\mathcal{F}$ induces a functor $\underline{\mathrm{Gp}}(\mathcal{C}) \longrightarrow \underline{\mathrm{Gp}}(\mathcal{D})$.

e) Deduce the same result as in the previous exercise.

a) definition snipped from wiki:

group object in C is an object G of C together with morphisms

- $m : G \times G \to G$ (thought of as the "group multiplication")
- $e : 1 \to G$ (thought of as the "inclusion of the identity element")
- $inv: G \to G$ (thought of as the "inversion operation")

such that the following properties (modeled on the group axioms – more precisely, on the definition of a group used in universal algebra) are satisfied

- m is associative, i.e. $m(m \times \mathrm{id}_G) = m(\mathrm{id}_G \times m)$ as morphisms $G \times G \times G \to G$, and where e.g. $m \times \mathrm{id}_G : G \times G \times G \to G \times G$; here we identify $G \times (G \times G)$ in a canonical manner with $(G \times G) \times G$.
- e is a two-sided unit of m, i.e. $m(\mathrm{id}_G \times e) = p_1$, where $p_1 : G \times 1 \to G$ is the canonical projection, and $m(e \times \mathrm{id}_G) = p_2$, where $p_2 : 1 \times G \to G$ is the canonical projection
- inv is a two-sided inverse for m, i.e. if $d : G \to G \times G$ is the diagonal map, and $e_G : G \to G$ is the composition of the unique morphism $G \to 1$ (also called the counit) with e, then $m(\mathrm{id}_G \times inv)\, d = e_G$ and $m(inv \times \mathrm{id}_G)\, d = e_G$.

Claim: group objects in $\mathcal{C}$ form a category, denoted by $\underline{Gp}(\mathcal{C})$

In $\underline{Gp}(\mathcal{C})$

objects: group objects of $\mathcal{C}$

morphism, $G \xrightarrow{f} H \xrightarrow{g} K$

f compatible with all the morphism defined above

explicitly:

$$\begin{tikzcd} G\times G \arrow[r, "f\times f"] \arrow[d, "m"'] \arrow[dr, phantom, "\circlearrowleft"] & H\times H \arrow[d, "m'"] \\ G \arrow[r, "f"] & H \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "="] \arrow[d, "e"'] \arrow[dr, phantom, "\circlearrowleft"] & 1 \arrow[d, "e'"] \\ G \arrow[r, "f"] & H \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "f"] \arrow[d, "inv"'] & H \arrow[d, "inv"] \\ G \arrow[r, "f"] & H \end{tikzcd}$$

only need to check $g \circ f$ satisfy the above restrictions

trivial

b) ① Group object in Category of Sets, $I = \{*\}$ one element sets

$G \in$ Sets

$m \quad G\times G \to G$

$e \quad 1 \to G \quad \Rightarrow \quad \exists\, id \in G \quad m(g\times id) = m(id \times g) = g$

$inv \quad G \to G \quad \forall\, g \in G \quad m(g \times inv(g)) = m(inv(g) \times g) = id_G$

$e_G : G \to G$

$g \mapsto id_G$

This are just the axioms for group

(2) Group object in the category of topological space

all the morphism m, e, inv are continuous map which make it to be a topological group

(3) $\mathcal{C}$ is the category of { open subsets of a fixed top sp X } with morphism "inclusion"

X is the terminal object in $\mathcal{C}$

in this case the product of two objects $A \times B \in obj(\mathcal{C})$

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] \arrow[d, dashed, "\exists! u"] & \\ A & A \times B \arrow[l] \arrow[r] & B \end{tikzcd}$$

$A \times B$ is in fact $A \cap B$ in $\mathcal{C}$

$m: A \times A \to A$

$inv: A \to A$

$e: X \to A$

$e \in Hom(X, A)$

$\Rightarrow$ A must be X itself.

where m, inv, e are all identity map on X

c) a group objects in the category of Groups

Gp (Groups)

$A \times B$ in Groups is the product group of A and B

terminal group is trivial group $\langle 1 \rangle$

$m: A \times A \to A$

inv: $A \to A$ these are all group morphisms

$e: \langle 1 \rangle \to A$ — amounts to mapping the id_A

$e : \{1\} \longrightarrow A$ — amounts to mapping to the id_A

e always exists

inv always exists

$$\begin{tikzcd} A\times A\times A \arrow[r, "m\times id_A"] \arrow[d, "id_A\times m"'] & A\times A \arrow[d, "m"] \\ A\times A \arrow[r, "m"] & A \end{tikzcd}$$

$(a,b,c) \longmapsto m(m(a,b),c)$

$\downarrow$

$m(a, m(b,c))$ // always true for m being a group homomorphism.

but for m being a group homomorphism

$g,h \in A\times A$ $\quad g=(g_1,g_2)$ $\quad h=(h_1,h_2)$

$gh = (g_1h_1, g_2h_2)$

$m(gh) = m(g)\cdot m(h)$

choose $g' = (g, id_A)$ $\quad h' = (id_A, h)$

$m(g,h) = m(gh) = m(g,id_A)\, m(h, id_A)$

$= g\cdot h$

$$\boxed{m((g,h)) = g\cdot h}$$

now look at $m(g,h)$ and $m(h,g)$

$m(g,h) = m\big((g,id)\cdot(id,h)\big) \leftarrow$ $\boxed{(g,id)\cdot(id,h) = (id,h)\cdot(g,id)}$

$= g\cdot h$

$m(g,h) = m\big((id,h)\cdot(g,id)\big)$

$= m(id,h)\cdot m(g,id)$

$= h\cdot g$

$\quad \| $

$m(h,g)$

$\Rightarrow h\cdot g = g\cdot h$

$\underline{Gp(\text{groups}) = \text{Abelian Groups}}$

d) $F: \mathcal{C} \longrightarrow \mathcal{D}$ is a functor preserving products and terminal objects

want:

Define a Functor $Gp(\mathcal{C}) \to Gp(\mathcal{D})$

want:

Define a functor $\underline{Gp}(\mathcal{C}) \to \underline{Gp}(\mathcal{D})$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "\underline{Gp}"'] & \mathcal{D} \arrow[d, "\underline{Gp}"] \\
\underline{Gp}(\mathcal{C}) \arrow[r, "\underline{Gp}(F)"] & \underline{Gp}(\mathcal{D})
\end{tikzcd}$$

$A, B, C \in obj(\mathcal{C})$

$f: A \to B \in Hom_{\mathcal{C}}(A,B)$ $\quad g \in Hom_{\mathcal{C}}(B,C)$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "F"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[r, "g"] \arrow[d, "F"] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[d, "F"] \\
F(A) \arrow[r, "F(f)"] & F(B) \arrow[r, "F(g)"] & F(C)
\end{tikzcd}$$

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if $A, B, C \in \underline{Gp}(\mathcal{C})$

$f \in Hom_{\underline{Gp}(\mathcal{C})}(A,B)$ $\quad g \in Hom_{\underline{Gp}(\mathcal{C})}(B,C)$

check ① $F(A), F(B), F(C) \in \underline{Gp}(\mathcal{D})$

② $F(f) \circ F(g) = F(fg)$ — trivial

for ① consider what it means by $F(m)$ $F(inv)$ $F(e)$

$$\begin{tikzcd}
A \times A \arrow[r, "m"] \arrow[d, "F"] & A \arrow[d] \\
F(A) \times F(A) \arrow[r, "F(m)"] & F(A)
\end{tikzcd}$$

F preserve products

$F(m): F(A) \times F(A) \to F(A)$

$(F(a), F(b)) \to F(m(a,b))$

automatically satisfy the following diagram

$$\begin{tikzcd}
A \times A \times A \arrow[dr, "m \times id_A"] \arrow[dd, "id_A \times m"'] \arrow[rr, "F \times F \times F"] & & F(A) \times F(A) \times F(A) \arrow[dr, "F(m) \times id_{F(A)}"] \arrow[dd, "id_{F(A)} \times F(m)" near start] & \\
& A \times A \arrow[rr, "F \times F" near start] \arrow[dd, "m" near start] & & F(A) \times F(A) \arrow[dd, "F(m)"] \\
A \times A \arrow[dr, "m"'] \arrow[rr, "F \times F" near start] & & F(A) \times F(A) \arrow[dr, "F(m)"] & \\
& A \arrow[rr, "F"] & & F(A)
\end{tikzcd}$$

similarly, we define

$F(inv): F(A) \to F(A)$

$F(a) \to F(inv(a))$

$$\begin{aligned} F(e): \; & F(1_{\mathcal{C}}) \to F(A) \\ & \;\; \| \\ & 1_{\mathcal{D}} \to F(A) \end{aligned}$$ — F preserve terminal object

$F(A)$ is still group object in $\mathcal{D}$

F induces a functor $\underline{Gp}(\mathcal{C}) \to \underline{Gp}(\mathcal{D})$

(e) $\pi_1(-,e)$ is a functor from {topological spaces} to {Groups}

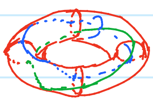
$\underline{Gp}(Top) = \{$Topological groups$\}$

$\underline{Gp}(Groups) = \{$Abelian groups$\}$

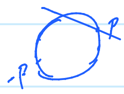
$\Rightarrow$ $\pi_1(G,e)$ is Abelian if G is a topological group

6. Let $X = \{(p, \cancel{p}) : p \neq -q\} \subseteq S^n \times S^n$ and consider the map $f : S^n \longrightarrow X$ sending $p \mapsto (p, p)$. Prove that it is a homotopy equivalence. [*Hint:* Define a map $X \longrightarrow S^n$ which you could not extend to the whole $S^n \times S^n$]

(p,q)



$g: X \to S^n$
$(p,q) \mapsto p$



$f \circ g : X \to S^n \to X$
$(p,q) \longmapsto (p,p)$

$g \circ f : S^n \to X \to S^n = id_{S^n}$
$p \mapsto (p,p) \mapsto p$

$$\boxed{X = \coprod_{p \in S^n} S^n \setminus \{-p\}}$$

$S^n \setminus \{-p\} \sim hep\ \{p\}$ $\forall p \in S^n$

$\exists\ h_p : S^n \setminus \{-p\} \times I \to S^n \setminus \{-p\}$
$h_p(x,t)$ s.t. $h_p(x,0) = x$
$h_p(x,1) = p \in S^n \setminus \{-p\}$

$h_X : X \times I \to X$
$(p,q) \times t \longmapsto$ --
$h_X((p,q),t) := (p, h_p(q,t))$

remain to check well-defined
h_X is continuous

$h_X((p,q),0) = (p,q)$ $\quad h_X((p,q),1) = (p,p)$

3.) c.) Let G be a group ⇝ terminal obj. is $\{1_G\}$

with $m: G\times G \to G$, $inv: G\to G$, $e:\{1_G\}\to G$ which satisfy axioms from ex. (a.)

compare ex. 2.a

$$m(a,b) = m(a\cdot_G 1_G, 1_G \cdot_G b) = m((a,1_G)\cdot_{G\times G}(1_G,b)) = m(a,1_G)\cdot m(1_G,b)$$

$$= m(a, e(1_G))\cdot m(e(1_G), b) = a\cdot_G b$$

$\uparrow$ $1_G = e(1_G)$ since e is a group homo.

compare ex. 2.b

$$a.b \overset{\text{above}}{=} m(a,b) = m(m(e(1),a), m(b,e(1))) \overset{\text{above}}{=} m(e(1),a)\cdot_G m(b,e(1))$$

$$= m((e(1),a)\cdot_{G\times G}(b,e(1))) = m(\underbrace{e(1)}_{=1}\cdot b,\ a\,\underbrace{e(1)}_{=1}) = m(b,a) \overset{\text{above}}{=} b.a$$

d.) $Gp(\mathcal{C}) \to Gp(\mathcal{C}')$ $(G, m, inv, e) \mapsto (\mathcal{F}(G), \mathcal{F}(m)\circ(\mathcal{F}(G)\times\mathcal{F}(G) \xrightarrow{\cong} \mathcal{F}(G\times G)), \mathcal{F}(inv), \mathcal{F}(e))$

$$\varphi \longmapsto \mathcal{F}(\varphi)$$

check axioms ...

e.) π: pointed top. spaces $\to$ groups, $(X,x)\mapsto \pi(X,x)$

$\overset{(d.)}{\rightsquigarrow}$ get function $\pi: Gp(Top) \longrightarrow Gp(Grp)$

i.e. $\pi_1(G,e)$ is a group object in $Gp(Grp)$ for top. groups G.

$\overset{(c.)}{\Longrightarrow}$ $\pi_1(G,e)$ is abelian.

4.) Consider the open cover $X = A\cup B\cup C$ $A\cap B =: C$

of the torus with two handles X, where A, B are choosen like this: (see picture)

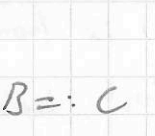
such that the base point p lies in A, B and C.

The pointed top. space (C,p) is homotopic to $(S^1,N) \Rightarrow \pi_1(C,p)\cong\pi_1(S^1,N)=\mathbb{Z}$

(A,p) and (B,p) are homeomorphic to the pointed torus with one hole $(\mathbb{T}\setminus\{pt\}, \text{basepoint})$

which has $\pi_1(\mathbb{T}\setminus\{pt\}, \text{basepoint}) = \mathbb{Z}*\mathbb{Z} = \langle a,b\rangle$

The generators a,b of $\mathbb{T}\setminus\{pt\}$ can be choosen like this:

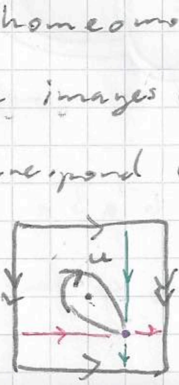


$\mathbb{T}\setminus\{pt\}$

The homeomorphisms can be choosen such that

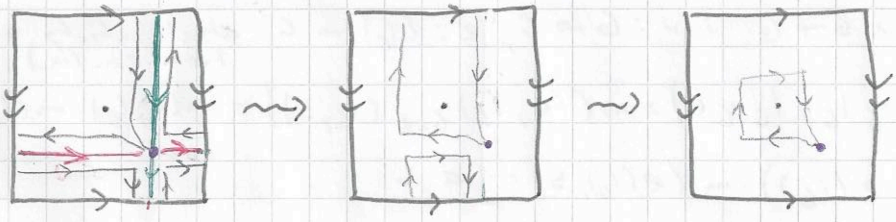
the images of the generator of $\pi_1(C)$ in $\pi_1(A)$ and $\pi_1(B)$

correspond to the same class in $\pi(\mathbb{T}\setminus\{pt\})$ with representant u: (see picture)



$\mathbb{T}\setminus\{pt\}$

The following pictures show that $u \sim_{htp} aba^{-1}b^{-1}$



Van Kampen implies that $\pi_1(X,x)$ is isomorphic to the colimit of the following diagram:

$$\begin{tikzcd} \mathbb{Z}*\mathbb{Z} & & \mathbb{Z}*\mathbb{Z} \\ & \mathbb{Z} \arrow[ul, "aba^{-1}b^{-1}" ', mapsto] \arrow[ur, "cdc^{-1}d^{-1}", mapsto] & \end{tikzcd}$$

$1 \mapsto aba^{-1}b^{-1}$, $1 \mapsto cdc^{-1}d^{-1}$

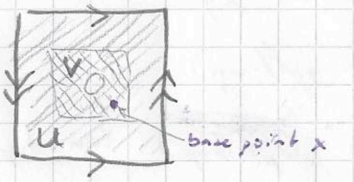
The formula for pushouts in Grp implies that

$$\pi_1(X,x) \cong \mathbb{Z}*\mathbb{Z}*\mathbb{Z}*\mathbb{Z} \big/ aba^{-1}b^{-1} \sim cdc^{-1}d^{-1}$$

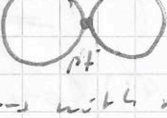
(i.e. $\sim$ is the smallest eq. rel with $aba^{-1}b^{-1} \sim cdc^{-1}d^{-1}$ s.t. $\mathbb{Z}*\mathbb{Z}*\mathbb{Z}*\mathbb{Z}/\sim$ is a group)

$$\cong \mathbb{Z}*\mathbb{Z}*\mathbb{Z}*\mathbb{Z} \big/ \mathrm{Norm}_{\mathbb{Z}*\mathbb{Z}*\mathbb{Z}*\mathbb{Z}}(\langle aba^{-1}b^{-1}dcd^{-1}c^{-1}\rangle)$$

5.) $X = U \cup V \cup W$, $W := U \cap V$ where U, V are chosen like this:

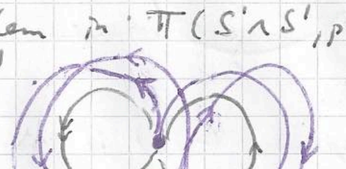


(U,x) is homotopic to the pointed top. space $(S^1 \wedge S^1, pt)$



which has fundamental group $\mathbb{Z}*\mathbb{Z}$ with generators with representants a and b like this

The generator of $\pi(W_{U\cap V}, p) = \mathbb{Z}$ gets mapped (via $W \hookrightarrow U$) to an element in $\pi_1(V, pt)$ which corresponds to an elem in $\pi(S^1 \wedge S^1, pt)$ with representant u (under the homotopy above) which looks like this:



as the following pictures show:

This representant u is therefore $u = b^{-1}aba$

Therefore $\pi_1(W,x) \xrightarrow{\pi_1(\subseteq)} \pi_1(U,x)$ is $\mathbb{Z} \to \mathbb{Z}*\mathbb{Z}$ $\quad 1 \mapsto b^{-1}aba$

Since V is simply connected it follows from a corollary of van-Kampen that $\pi_1(X,x) = \mathbb{Z}*\mathbb{Z} \big/ \mathrm{Norm}_{\mathbb{Z}*\mathbb{Z}}(\langle b^{-1}aba \rangle)$

7.1 d.) (Idea: Try to generalize ex 6. b. and c of series 9 of Alg. Top. I.) ⑥

Let $F: I \to C$ be a functor. Let A be the class of morphisms of I.

Since by assumption, arbitrary coproducts exist, both

$\coprod_{\alpha\in A} F(\mathrm{dom}\,\alpha)$ and $\coprod_{i\in I} F(i)$ exist.

For every $\alpha\in A$ we get two morphisms $F(\alpha): F(\mathrm{dom}(\alpha)) \to F(\mathrm{codom}(\alpha))$

and $id_{F(\mathrm{dom}\,\alpha)}: F(\mathrm{dom}(\alpha)) \to F(\mathrm{dom}(\alpha))$.

If we compose these with the inclusion morphisms of $\coprod_{i\in I} F(i)$

we get two morphisms $F(\mathrm{dom}(\alpha)) \to \coprod_{i\in I} F(i)$ and $F(\mathrm{dom}(\alpha)) \to \coprod_{i\in I} F(i)$

for every $\alpha\in A$.

If we apply the U.P. of $\coprod_{\alpha\in A} F(\mathrm{dom}\,\alpha)$ to these two

collections of morphisms we get two morphisms

$f, g \quad \coprod_{\alpha\in A} F(\mathrm{dom}\,\alpha) \rightrightarrows \coprod_{i\in I} F(i)$

Since by assumption arbitrary coequalizers exist, there

is a coequalizer C with a coeq. morph $c: \coprod F(i) \to C$

of the two morphisms f, g.

If we compose the coeq. morph with the inclusion morphisms

of $\coprod F$ we get morphisms $\varphi_i: F(i) \to C$.

These satisfy the U.P. of the colimit:

For any $\alpha: i\to j$:

$\varphi_j \circ F(\alpha) = c\circ(incl_j: F(j) \to \coprod_i F(i)) \circ F(\alpha)$ $\quad c\circ f = c\circ g$

$\quad = c\circ f\circ(incl_\alpha: F(i) \to \coprod_\alpha F(\mathrm{dom}\,\alpha)) = c\circ g\circ(incl_\alpha)$

$\quad = c\circ(incl_i) = \varphi_i$

For any other C' with $\varphi_i': F(i) \to C'$ s.t. $\forall\alpha: i\to j: \varphi_j'\circ F(\alpha) = \varphi_i'$

we can define a morph. $c': \coprod F \to C'$ (U.P. of coprod.)

for every $\alpha: i\to j$: $c'\circ f\circ(incl_\alpha) = c'\circ(incl_j)\circ F(\alpha) = \varphi_j'\circ F(\alpha) = \varphi_i'$

$\quad = c'\circ incl_i = c'\circ g\circ(incl_\alpha)$

$\xRightarrow{\text{U.P. coprod}} c'\circ f = c'\circ g$

continued on p. 7

7.) d.) continued.)

Therefore the U.P. of the coeq. gives a unique morphism

$d: C \to C'$ with $d \circ c = c'$.

By the U.P. of $\coprod_i F(i)$ this is equivalent to $\forall i \in I$ $d \circ \varphi_i = \varphi_i'$

Therefore the U.P. of the colim is satisfied.

7.) e.) Let $\mathcal{C}$ be a cat. with all products and all products and equivalisers.

Let $G: J \to \mathcal{C}^{op}$ be a functor, where J is a discrete category (only morph are identities)

By assumption a product $\prod_{j \in J^{op}} G^{op}(j)$ of the opposite functor

$G^{op}: J^{op} \to \mathcal{C}$ exists, with morphisms $\psi_j: \prod_{j \in J^{op}} G^{op}(j) \to G^{op}(j)$

These morph. in $\mathcal{C}$ are morphisms in $\mathcal{C}^{op}$ $\psi_j: G(j) \to \prod_{j \in J^{op}} G^{op}(j)$

The U.P of the product $\prod_{j \in J^{op}} G^{op}(j)$ with ψ_j in $\mathcal{C}$

translates in to the U.P. of the coproduct

$\coprod_{j \in J^{op}} G(j) = \prod_{j \in J^{op}} G^{op}(j)$ with morph $\varphi_j := \psi_j$ for $j \in J$ in $\mathcal{C}^{op}$

(Indeed: let $\varphi_j': G(j) \to C'$ be morphisms in $\mathcal{C}^{op}$

$\rightsquigarrow$ morph. $\varphi_j': C' \to G(j) = G^{op}(j)$ in $\mathcal{C}$

$\rightsquigarrow$ get unique $\delta: C' \to \prod_{j \in J^{op}} G^{op}(j)$ such that

for all $j \in J$ $\varphi_j' = \underbrace{\psi_j}_{=\varphi_j} \circ \delta$ in $\mathcal{C}$

This condition $\forall j \in J$ $\varphi_j' = \varphi_j \circ \delta$ in $\mathcal{C}$ is eq. to $\forall j \in J$ $\varphi_j' = \delta \circ \varphi_j$ in $\mathcal{C}^{op}$

therefore $\delta: \coprod_{j \in J} G(j) = \prod_{j \in J^{op}} G^{op}(j) \to C'$ in $\mathcal{C}^{op}$ is a unique

morph with $\forall j \in J$ $\varphi_j = \delta \circ \varphi_j$)

$\Rightarrow G$ has a coproduct in $\mathcal{C}^{op}$ $\xRightarrow{G \text{ arbitrary}}$ $\mathcal{C}^{op}$ has all coproducts

Analogously $\mathcal{C}^{op}$ has all coequalisers

$\}\xRightarrow{(d.)}$ $\mathcal{C}^{op}$ has all colimits.

Let now $F: I \to \mathcal{C}$ be a functor then F^{op} has a colimit in $\mathcal{C}^{op}$.

Similarly than above we see, that this colimit satisfies the

U.P. of the limit of F in $\mathcal{C}$. $\xrightarrow{F \text{ arbitrary}}$ $\mathcal{C}$ is complete.

q.e.d.