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AT II - Exercise Sheet 10

Def A subcomplex of X is a closed subspace $Y \subset X$ which is a union of cells of X.

Exercise 1 X CW complex, A compact subset of X.

Claim A is contained in a finite subcomplex of X.

Proof ① A meets only finitely many cells of X.

Pf Suppose by contradiction that there is an infinite sequence of points $x_i \in A$ all lying in distinct cells.

Then the set $S = \{x_1, x_2, \dots\}$ is closed in X.

Indeed: Assume $S \cap X_{n-1}$ is closed in X_{n-1}. Then for each cell of dim n e^n_α of X, $\varphi_\alpha^{-1}(S)$ [$\varphi_\alpha^{-1}(S \cap X_{n-1})$] is closed in $\partial D^n_\alpha = S^{n-1}_\alpha$ ($\varphi_\alpha : S^{n-1}_\alpha \to X_{n-1}$ attaching map)

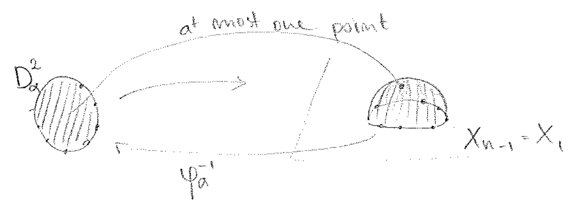
[$\Phi_\alpha : D^n_\alpha \hookrightarrow X$] and $\Phi_\alpha^{-1}(S)$ consists of at most one more point in D^n_α (b/c just one point of S in e^n_α)

$\xrightarrow{\text{adding a point doesn't change closeness}}$ $\Phi_\alpha^{-1}(S)$ is closed in D^α_n.

So $\Phi_\alpha^{-1}(S)$ is closed in D^α_n $\forall \alpha$ $\xrightarrow{\text{Rmk about topology on CW complexes}}$ $S \cap X_n$ is closed in X_n.

[Rmk about topology: $\Phi_\alpha : D^n_\alpha \hookrightarrow X_{n-1} \sqcup \bigsqcup_\alpha D^n_\alpha \xrightarrow[\text{quot}]{\pi_n} X_n \hookrightarrow X$. Suppose that for $U \subset X$, $\Phi_\alpha^{-1}(U)$ is open in D^n_α $\forall \alpha, \forall n$. Then $U \cap X_n$ is open in X_n $\forall n$ i.e. U is open in X. By induction:
$n=0$: $U \cap X_0$ is open in X_0 b/c discrete.
$n-1 \to n$: Suppose $U \cap X_{n-1}$ is open in X_{n-1}, and $\Phi_\alpha^{-1}(U)$ is open in D^n_α. Then $\underbrace{\bigsqcup_\alpha \Phi_\alpha^{-1}(U) \sqcup (U \cap X_{n-1})}_{= \pi_n^{-1}(U \cap X_n)}$ is open in $\bigsqcup_\alpha D^n_\alpha \sqcup X_{n-1}$ $\xrightarrow[\text{quot. topology}]{\text{Def of}}$ $U \cap X_n$ open in X_n.]

Idea:



So the set S is closed $\subseteq A$, hence compact. Since it is discrete, it is finite. ↯ $\square$①

b/c each point of S is in the interior of a cell, which doesn't intersect the interior of other cells.

② The union of finitely many cells is contained in a finite subcomplex of X.

Proof Since the union of finitely many finite subcomplexes is still a finite subcomplex, it suffices to show that a single cell is contained in a finite subcomplex of X.

Induction on the dimension of the cell: $n = 1$: cell = open unit interval. We attach the unit interval to one or two points $\to$ finite subcomplex of X $\supset$ open unit interval

Suppose the assertion holds for cells of dim $< n$.

Let e^n be a cell of dim n. Then $\overline{e^n} \setminus e^n$ is compact so by ① is contained in finitely many cells which must have dim $< n$ by def of CW-complex. By induction each of these is contained in a finite subcomplex. The union of these is a finite subcomplex Y containing $\overline{e^n} \setminus e^n$. Then $Y \cup \overline{e^n}$ is a finite subcomplex containing e^n.

□ Ex 1

Exercise 2 Let X be a CW complex with filtration $X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots$

Claim Let $k \geq 0$. Then $\pi_k(X_k) \longrightarrow \pi_k(X_{k+1})$ is surjective

and $\pi_k(X_n) \longrightarrow \pi_k(X_{n+1})$ is an isom. for $n > k$.

Proof : ① $\pi_k(X_n) \to \pi_k(X_{n+1})$ is surjective $\forall n \geq k$.

Let $\alpha : S^k \to X_{n+1}$ represent an element of $\pi_k(X_{n+1})$.

By cell-approx. for maps, α is homotopic to a map with image in X_k (S^k is k-dim'l complex ($S^k = D^k/\partial D^k$))

Since $n \geq k$, the claim follows.

② $\pi_k(X_n) \longrightarrow \pi_k(X_{n+1})$ is injective $\forall n > k$.

Let $\alpha, \beta : S^k \to X_n$ in $\pi_k(X_n)$ with same image in $\pi_k(X_n)$, that is

there is a homotopy $h : \underbrace{S^k \times I}_{\text{CW-complex of dim } k+1} \to X_{n+1}$ between α and β.

$\xRightarrow[(k+1 \leq n)]{\text{cellular approx. for maps}}$ h is homotopic to a map $\tilde{h} : S^k \times I \to X_n$.

Let $\tilde{\alpha} = \tilde{h}(\cdot, 0) : S^k \to X_n$

$\tilde{\beta} = \tilde{h}(\cdot, 1) : S^k \to X_n$

Then
- $\tilde{\alpha}$ and $\tilde{\beta}$ are homotopic in $\pi_k(X_n)$ via $\tilde{h}$.
- $\tilde{\alpha}$ is homotopic to α in $\pi_k(X_{n+1})$:

  Let $K : (S^k \times I) \times I \longrightarrow X_{n+1}$ be a homotopy between h and $\tilde{h}$,

  st $K(\cdot, 0) = h(\cdot)$, $K(\cdot, 1) = \tilde{h}(\cdot)$.

  Then $K(\cdot, 0, 0) = h(\cdot, 0) = \alpha(\cdot)$

  $K(\cdot, 0, 1) = \tilde{h}(\cdot, 0) = \tilde{\alpha}(\cdot)$

  So that $K(\cdot, 0, \cdot) : S^k \times I \to X_{n+1}$ is a homotopy between $\alpha, \tilde{\alpha}$.
- Same for $\tilde{\beta}, \beta$.

$\Longrightarrow$ $\tilde{\alpha} = \tilde{\beta}$ in $\pi_k(X_n)$ and $\tilde{\alpha} = \alpha$, $\tilde{\beta} = \beta$ in $\pi_k(X_{n+1})$ showing injectivity.

□ Ex 2

Exercise 3

Let X be a CW-complex st X is the union of subcomplexes $X_1 \subseteq X_2 \subseteq \dots$ st each inclusion $X_i \hookrightarrow X_{i+1}$ is nullhomotopic.

Claim X is contractible

Proof The idea is to show that $\pi_n(X, x_0) = 0 \quad \forall n$ and then to use Whitehead's Theorem.

So let $\alpha : S^n \to X$ represent an element of $\pi_n(X, x_0)$.

Then $\alpha(S^n)$ is compact $\overset{\text{Ex 1}}{\Longrightarrow}$ $\alpha(S^n) \subseteq X_k$ for some k.

By assumption $i_k : X_k \to X_{k+1}$ is nullhomotopic

$\Longrightarrow$ $i_k \circ \alpha : S^n \to X_{k+1}$ is nullhomotopic, hence also $\alpha = i \circ i_k \circ \alpha : S^n \to X$ is

(where $i : X_{k+1} \hookrightarrow X$), i.e. $\alpha = 0$ in $\pi_n(X, x_0)$.

This shows that $\pi_n(X, x_0) = 0$.

Now, let $g : X \to \{x_0\}$. Since $\pi_n(X, x_0) = 0 = \pi_n(\{x_0\}) \quad \forall n$, the induced map $g_* : \pi_n(X, x_0) \to \pi_n(\{x_0\})$ is an isomorphism $\forall n$.

$\overset{\text{Whitehead's thm}}{\Longrightarrow}$ g is a homotopy equivalence.

□ Ex 3.

Exercise 4

a) How to define the infinite sphere S^∞ as a CW complex ?

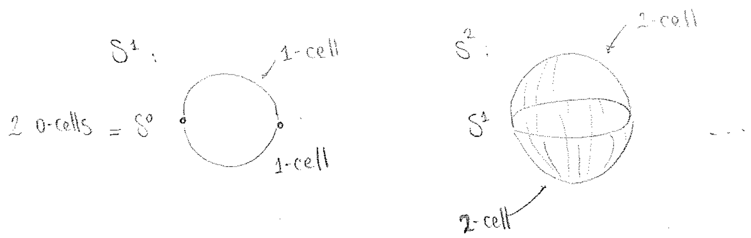
$$S^\infty = \bigcup_{n \in \mathbb{N}} S^n .$$

We usual see $S^0 \subset S^1 \subset S^2 \subset \dots$ as: $S^n = \{ (x_1, \dots, x_n, 0) \mid x_1^2 + \dots + x_n^2 = 1 \} \subset S^{n+1}$

Recall that S^n is a CW-complex given by a 0-cell with an n-cell attached.

Then $S^{n-1} \subset S^n$ as viewed above is not a subcomplex (= closed subspace $A \subseteq X$ which is a union of cells of X)

Inductively : S^n is obtained from the equatorial S^{n-1} by attaching two n-cells = the components of $S^n \setminus S^{n-1}$.



$\Rightarrow$ S^n is a CW-complex with S^{n-1} as subcomplex. (Still we have $S^{n-1} \subset S^n$ as equator. We just don't regard S^n as a CW complex in the usual way)

b) S^∞ is contractible.

Proof The inclusion $i : S^n \hookrightarrow S^{n+1}$ is null-homotopic.
$x \longmapsto (x, 0)$

A homotopy between i & const map is given by $h : S^n \times I \longrightarrow S^{n+1}$
$(x, t) \longmapsto (\sqrt{1-t^2}\, x, t)$

(notice : $\| (\sqrt{1-t^2}\, x, t) \|^2 = (1-t^2) \underbrace{\|x\|^2}_{=1} + t^2 = 1 - t^2 + t^2 = 1$ so that it is well-def)

$h(x, 0) = (x, 0) = i(x)$
$h(x, 1) = (0, 1) = \text{const}$.

$\xRightarrow{\text{Ex 3}}$ S^∞ is contractible.

Exercise 5 An n-connected, n-dimensional CW complex is contractible.

Recall • X n-dim'l : $X_0 \subseteq X_1 \subseteq \dots \subseteq X_n = X$ filtration.

• X n-connected: $X \to pt$ induces an isomorphism $\pi_k(X,x_0) \to \pi_k(pt)$ $\forall\, 0 \le k \le n$.

Proof Apply cellular approx. of spaces to $(X,\{x_0\})$ to get a CW pair $(Z,\{x_0\})$ and a map $f: Z \to X$ st $f_*: \pi_k(Z,x_0) \to \pi_k(X,x_0)$ is an iso $\forall k > n$ and $\pi_k(Z,x_0) = 0$ $\forall k \le n$. Moreover Z is obtained by attaching cells of dim $> n$ to $\{x_0\}$.

By the first two properties and the fact that X is n-connected we have that f is a weak equivalence and therefore a homotopy equivalence. (Whitehead)

Let $g: X \to Z$ be a homotopy inverse of f.

By cellular approx. for maps g is homotopic to a map with image in $\{x_0\}$ since X is n-dimensional and Z has one 0-cell and all others are $> n$ dim'l.

$\Rightarrow$ $id_X \sim f \circ g$ is homotopic to a constant map, so X is contractible.

□

Exercise 6 X, Y homotopy equivalent CW complexes, both without $(n+1)$-cells.

Claim The n-skeletons of X and Y are also homotopy equivalent.

Proof The n-skeletons of X, Y are X_n, Y_n.

By assumption we have $f: X \to Y$ and $g: Y \to X$ homotopy inverses.

By cellular approx. for maps we can assume that are both cellular maps.

Let $f_n: X_n \to Y_n$, $g_n: Y_n \to X_n$ be the restrictions to the n-skeleton.

We claim that they are homotopy inverses to each other.

Let $h: X \times I \to X$ be a homotopy between $g \circ f$ and id_X.

By cellular approx. we may assume that h is cellular to get a map $h_n: X_n \times I \to X_{n+1} = X_n$ since X has no $(n+1)$-cells.

We have : For $x \in X_n$ we have
$$h_n(x,0) = h(x,0) = g \circ f(x) = g \circ f_n(x) = g_n \circ f_n(x)$$
$$h_n(x,1) = h(x,1) = id_X(x) = id_{X_n}(x).$$

So $g_n \circ f_n \sim id_{X_n}$.

Similarly, $f_n \circ g_n \sim id_{Y_n}$.

□ Ex 6.