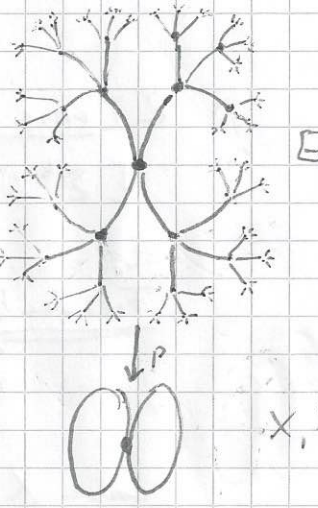
# Algebraische Topologie Serie 12

1.)(a) <u>case n=1</u>: Consider $Z_1 := \mathbb{Z}*\mathbb{Z} \amalg \coprod_{g\in\mathbb{Z}*\mathbb{Z}} ([0,1] \amalg [0,1])$ ← denoted by $I_g^b$ in $Z_1$, ← denoted by $I_g^a$ in $Z_1$

Let the eq. rel. $\sim$ on $Z_1$ be

gen. by $I_g^a \ni 0 \sim g$ , $I_g^b \ni 0 \sim g$

$I_g^a \ni 1 \sim [a]g$ $\quad I_g^b \ni 1 \sim [b]g$




$E_1 := Z_1/\sim$

$\mathbb{Z}*\mathbb{Z} \longrightarrow$ basepoint $\hookrightarrow S^1 \vee S^1$ and

$I_g^a \xrightarrow{e^{2\pi i t}} S^1 \xhookrightarrow{incl_1} S^1 \vee S^1$ and

$I_g^b \xrightarrow{e^{2\pi i t}} S^1 \xhookrightarrow{incl_2} S^1 \vee S^1$ induces a covering

$E_1 \xrightarrow{p} S^1 \vee S^1$

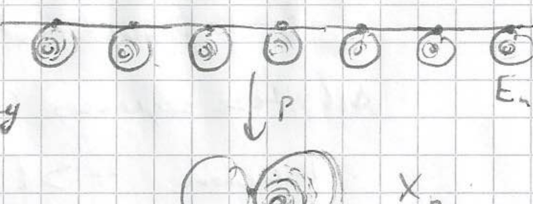
$E_1$ is simply connected $\Rightarrow$ universal covering.

$\pi_1(X_1) = \mathbb{Z}*\mathbb{Z}$ acts on $p^{-1}(\text{basepoint}) = \mathbb{Z}*\mathbb{Z} \subset E_1$

by translation: $g \longmapsto hg$ for $h \in \pi_1(X_1)$

<u>case $n \geq 2$</u>: $Z_n := \left(\coprod_{k\in\mathbb{Z}} S^n\right) \amalg \mathbb{R}$ ← denoted by $S_k^n$ in $Z$

$E_n = Z_n/\sim$ where $\sim$ identifies the basepoint of $S_k^n$ with $k \in \mathbb{Z}$.




$S_k^n \xrightarrow{\sim} S^n$ and

$\mathbb{R} \xrightarrow{e^{2\pi i t}} S^1$ induce a covering

$E_n \xrightarrow{p} S^1 \vee S^n$.

$E_n$ simply connected $\Longrightarrow$ universal covering.

Since $\pi_1(S^n) = 0$ we can find a htp from every path in $S^1 \vee S^n$

to a path with image in $S^1$ $\xrightarrow[\text{lemma}]{\text{compression}}$ $\pi_1(S^1 \vee S^n, S^1) = 0$

It follows with the exact seq. corr. to the pair $(S^1 \vee S^n, S^1)$

that $\pi_1(S^1 \hookrightarrow S^1 \vee S^n) : \underbrace{\pi_1(S^1)}_{\cong \mathbb{Z}} \xrightarrow{\sim} \pi_1(S^1 \vee S^n)$ is an isom.

$\pi_1(S^1 \vee S^n)$ acts on $p^{-1}(\text{basepoint}) = \mathbb{Z} \subset \mathbb{R} \subset E_n$

by translation $z \longmapsto x + z$ for $x \in \mathbb{Z} \cong \pi_1(S^1 \vee S^n)$

b.) For $n \geq 2$ there is a htp. eq. $E_n \simeq_{htp} \bigvee_{\mathbb{Z}} S^n$ ← Bracket means "elem. in $\mathbb{Z}^{\mathbb{Z}}$ which have almost all coordinates $= 0$."

$\Rightarrow \pi_n(E_n) \cong \pi_n(\bigvee_{\mathbb{Z}} S^n) \xrightarrow[\text{ex. 2}]{\text{Sheet 11}} (\pi_n(S^n))^{(\mathbb{Z})} = \mathbb{Z}^{(\mathbb{Z})}$

$\mathbb{Z}^{(\mathbb{Z})} \cong \mathbb{Z}[t, t^{-1}], \quad (a_i)_{i \in \mathbb{Z}} \longmapsto \sum_{i \in \mathbb{Z}} a_i t^i$

$\pi_1(X_n)$ acts on $\mathbb{Z}[t, t^{-1}]$ by

$$f(t) \longmapsto t^k \cdot f(t) \quad \text{for } k \in \mathbb{Z} \cong \pi_1(X_n)$$

2.) <u>case $X = S^n$, $Y$ connected:</u>

$[\Sigma^k S^n, \Sigma^k Y] = \pi_{n+k}(\Sigma^k Y)$ and $\Sigma^k Y$ is $k$-connected

$\overset{\text{Freudenthal}}{\Longrightarrow}$ $\pi_{n+k}(\Sigma^k Y) \longrightarrow \pi_{n+k+1}(\Sigma^{k+1} Y)$

$[\Sigma^k S^n, \Sigma^k Y] \xrightarrow{\Sigma} [\Sigma^{k+1} S^n, \Sigma^{k+1} Y]$

is an isomorphism for $n+k \leq 2k \Leftrightarrow n \leq k$

<u>case $X = S^n$, $Y$ maybe not connected</u>

$Y = \coprod_{i \in I} Y_i$ decomposition into conn. components $Y_i$

$\Sigma^k Y = \Sigma^k \coprod_{i \in I} Y_i = \Sigma^{k-1} \coprod_{i \in I} \Sigma Y_i = \ldots = \coprod_{i \in I} \Sigma^k Y_i$

$[\Sigma^k X, \underbrace{\Sigma^k Y}_{\coprod_{i \in I} \Sigma^k Y_i}] = \coprod_{i \in I} [\Sigma^k X, \Sigma^k Y_i]$ ← $\Sigma^k X$ connected bec. $X = S^n$ connected

Therefore the sequence $\ldots \to [\Sigma^k X, \Sigma^k Y] \to [\Sigma^{k+1} X, \Sigma^{k+1} Y] \to \ldots$

is the coproduct of $\ldots \to [\Sigma^k X, \Sigma^k Y_i] \to [\Sigma^{k+1} X, \Sigma^{k+1} Y_i] \to \ldots$ for $i \in I$

All these sequences for $i \in I$ become stationary for $n \leq k$ by the case above.

Therefore $\ldots \to [\Sigma^k X, \Sigma^k Y] \to [\Sigma^{k+1} X, \Sigma^{k+1} Y] \to \ldots$ becomes

stationary for for $n \leq k$.

<u>case $X$ finite dim. CW-complex, $Y$ maybe not connected</u>

Since $X$ is a fin. dim. CW-complex, we have

$X = \operatorname{colim}_{j \in J} S^{n_j}$, $n_j \in \mathbb{Z}^{\geq 0}$ for $j \in J$ with some $N \in \mathbb{Z}^{\geq 0}$ s.t. $\forall j$ $n_j \leq N$

Since $\Sigma$ is left adjoint to $\Omega$ it commutes with

the colimit: $\Sigma^k X = \Sigma^k \operatorname{colim}_{j \in J} S^{n_j} = \Sigma^{k-1} \operatorname{colim}_{j \in J} \Sigma S^{n_j} = \ldots = \operatorname{colim}_{j \in J} \Sigma^k S^{n_j}$

$[\underbrace{\Sigma^k X}_{\operatorname{colim}_{j \in J} \Sigma^k S^{n_j}}, \Sigma^k Y] = \lim_{j \in J} [\Sigma^k S^{n_j}, \Sigma^k Y]$ ← representable functors take colimits to limits

2.) continued.)

Therefore the sequence $\dots \to [\Sigma^n X, \Sigma^n Y] \to [\Sigma^{n+1} X, \Sigma^{n+1} Y] \to \dots$

is the limit of the sequences

$\dots \to [\Sigma^n S^{n_j}, \Sigma^n Y] \to [\Sigma^{n+1} S^{n_j}, \Sigma^{n+1} Y] \to \dots$ for $j \in J$.

All of these sequences become stationary for $N \le n \Rightarrow \forall_j \; n_j \le n$

by the previous case.

Therefore $\dots \to [\Sigma^n X, \Sigma^n Y] \to [\Sigma^{n+1} X, \Sigma^{n+1} Y] \to \dots$

becomes stationary for $N \le n$. qed.

3.) From the definition of $f+g$ it follows that

$*$. $$f+g = \mathrm{fold}_K \circ (f \vee g) \circ \alpha \quad \text{where} \quad \alpha : \begin{array}{l} \Sigma X \to (\Sigma X) \vee (\Sigma X) \\ [t,x] \mapsto \begin{cases} \mathrm{incl}_1([2t,x]) & t \le \frac{1}{2} \\ \mathrm{incl}_2([2t-1,x]) & t > \frac{1}{2} \end{cases} \end{array}$$

Since $E$ is additive we have

$**$ $$\begin{pmatrix} E(f) & 0 \\ 0 & E(g) \end{pmatrix} \circ \begin{pmatrix} E(\mathrm{incl}_1 : K \to K \vee K) \\ E(\mathrm{incl}_2 : K \to K \vee K) \end{pmatrix} = \begin{pmatrix} E(\mathrm{incl}_1 : \Sigma X \to \Sigma X \vee \Sigma X) \\ E(\mathrm{incl}_2 : \Sigma X \to \Sigma X \vee \Sigma X) \end{pmatrix} \circ E(f \vee g)$$

$***$ The isomorphism $\begin{pmatrix} E(\mathrm{incl}_1 : \Sigma X \to \Sigma X \vee \Sigma X) \\ E(\mathrm{incl}_2 : \Sigma X \to \Sigma X \vee \Sigma X) \end{pmatrix}$ has inverse $\big(\underbrace{E((id_{\Sigma X}, const))}, E((const, id_{\Sigma X}))\big)$

By additivity of E.

(This means the maps $\Sigma X \vee \Sigma X \to \Sigma X$ induced by $id_{\Sigma X} : \Sigma X \to \Sigma X$ and $const : \Sigma X \to \Sigma X$)

It follows:

$$E(f+g) \overset{*}{=} E(\alpha) \circ E(f \vee g) \circ E(\mathrm{fold}_K) \overset{**,\,***}{=\!=}$$

$$E(\alpha) \circ \big(E((id_{\Sigma X}, const)), E((const, id_{\Sigma X}))\big) \circ \begin{pmatrix} E(f) & 0 \\ 0 & E(g) \end{pmatrix} \circ \begin{pmatrix} E(\mathrm{incl}_1 : K \to K \vee K) \\ E(\mathrm{incl}_2 : K \to K \vee K) \end{pmatrix} \circ E(\mathrm{fold}_K)$$

$$= \big(E(\underbrace{(id_{\Sigma X}, const) \circ \alpha}), E(\underbrace{(const, id_{\Sigma X}) \circ \alpha}_{\simeq_{htp}\, id_{\Sigma X}})\big) \circ \begin{pmatrix} E(f) & \\ & E(g) \end{pmatrix} \circ \begin{pmatrix} E(\underbrace{\mathrm{fold}_K \circ \mathrm{incl}_1}_{=id}) \\ E(\underbrace{\mathrm{fold}_K \circ \mathrm{incl}_2}_{=id}) \end{pmatrix}$$

$$\left( [t,x] \mapsto \begin{cases} [2t,x] & t \le \frac{1}{2} \\ \text{base point} & t \ge \frac{1}{2} \end{cases} \right) \simeq_{htp} id_{\Sigma X}$$

$$= (id, id) \begin{pmatrix} E(f) & \\ & E(g) \end{pmatrix} \begin{pmatrix} id \\ id \end{pmatrix} = E(f) + E(g)$$

qed