Algebraische Topologie II Serie 2

(1)

1.) a.) Since p is a covering we get for every point $h \in H$ a neighbourhood $V \ni h$ which is homeomorphic to some neighbourhood $U \ni p(h)$.

Since G is locally path connected, $U \underset{G}{\cong} V$ is loc. path-connected.

Therefore every point in H has a loc. path. conn. n.b.h., hence H is loc. path conn.

Since H is connected it follows that it is path connected.

Let $m: G \times G \to G$ be the mult. of G. $\Rightarrow m$ cont.

p cont. $\Rightarrow$ $p \times p : H \times H \to G \times G$ is cont.

$\Rightarrow m' := m \circ (p \times p)$ is cont. $H \times H \longrightarrow G$

$p(f) = e \Rightarrow m'(f,f) = e$ $\qquad$ H path conn. $\Rightarrow H \times H$ path conn.

Claim: $m'_*(\pi_1(H \times H, (f,f)) \subset p_*(\pi_1(H, f))$

proof: Every elem. in $\pi_1(H \times H, (f,f))$ can be written as $[t \mapsto (\alpha(t), \beta(t))]$ for $[\alpha], [\beta] \in \pi_1(H, f)$

Then $m'_*([t \mapsto (\alpha(t), \beta(t))]) = [t \mapsto m(p(\alpha(t)), p(\beta(t)))]$

$\overset{\text{Sheet 1 ex. 2. a.}}{=} [p \circ \alpha * p \circ \beta] = p_*([\alpha]) \cdot p_*([\beta]) = p_*([\alpha] \cdot [\beta])$ qed (Claim)

For these reasons I can apply the fundamental theorem of covering spaces to get a unique $m'': H \times H \to H$ with

$$\begin{tikzcd} H \times H \arrow[r, dashed, "m''"] \arrow[d, "p \times p"'] \arrow[dr, "m' := m \circ (p \times p)"] & H \arrow[d, "p"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd}$$

(f,f) $\qquad$ e

$p \circ m'' = m' \overset{def}{=} m \circ (p \times p)$ $(*)$ and $m''(f,f) = f$ $(**)$

m'' respects the neutral elem. f:

Let $a \in H$ $\rightsquigarrow$ get a path α from f to a.

The paths $t \mapsto m''(f, \alpha(t))$ and α both start at f bec. of $**$.

$p(m''(f, \alpha(t))) \overset{*}{=} m(\underset{=e}{p(f)}, p(\alpha(t))) = p(\alpha(t))$

Since p is a covering we have $m''(f, \alpha(t)) = \alpha(t) \overset{t:=1}{\Longrightarrow} m''(f, a) = a$

Analogous we get $m''(a, f) = a$.

m'' is associative:

Let $a, b, c \in H$ $\rightsquigarrow$ get a path β from f to b.

Since m'' respects f the paths $t \mapsto m''(m''(a, \beta(t)), c)$ and $t \mapsto m''(a, m''(\beta(t), c))$ both start at $m''(a, c)$.

$$p(m''(m''(a,\beta(t)),c)) \overset{\text{apply } * \text{ two times}}{=} m(m(p(a),p(\beta(t))),p(c))$$

$$\overset{m \text{ assoc.}}{=} m(p(a),m(p(\beta(t)),p(c))) \overset{\text{apply } * \text{ two times}}{=} p(m''(a,m''(\beta(t),c)))$$

$$\overset{p \text{ covering}}{\Longrightarrow} m''(m''(a,\beta(t)),c) = m''(a,m''(\beta(t),c)) \overset{t:=1}{\Rightarrow} m''(m''(a,b),c)=m''(a,m''(b,c))$$

<u>inverse elements:</u>

$i: G \to G,\ g \mapsto g^{-1}$ is cont. since G is a top. group.

Define $i' := i \circ p$.

Similarly to how we constructed m'' we can construct an $i'': H \to H$ with $p \circ i'' = i'$ $(**)$ and $i''(f) = e$ $(**)$

$$\begin{tikzcd} H \arrow[r, dotted, "i''"] \arrow[d, "p"'] \arrow[dr, "i':=i\circ p"] & H \arrow[d, "p"] \\ G \arrow[r, "i"'] & G \end{tikzcd}$$

(For this show $i'_*(\pi_1(H,f)) \subset p_*(\pi_1(H,f))$ using the fact that $[\alpha]^{-1} = [t \mapsto i(\alpha(t))]$)

$\rightsquigarrow$ get a cont. $i'': H \to H$

For $a \in H$ find a path α from f to a.

Both the paths $m''(\alpha(t), i''(\alpha(t)))$ and $c_f: t \mapsto f$ start at f

$$p(m''(\alpha(t), i''(\alpha(t)))) \overset{*}{=} m(p(\alpha(t)), p(i''(\alpha(t)))) \overset{**}{=} m(p(\alpha(t)), i(p(\alpha(t)))) = e = p(f)$$

$$\overset{p \text{ covering}}{\Longrightarrow} m''(\alpha(t), i''(\alpha(t))) = f \Rightarrow m''(a, i''(a)) = f$$

Analogous $m''(i''(a), a) = f$.

In summary (H, m'', i'', f) is a topological group.

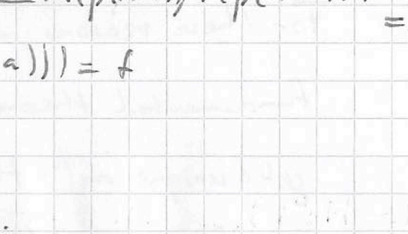
p is a group homo bec of $*$ and $p(f) = e$.

b.) We already saw that H is a top. group.

Let G be abelian. For $a, b \in H$ take a path α from f to a.

The same argument that we already made 3 times now with the paths $t \mapsto m''(\alpha(t), b)$ and $t \mapsto m''(b, \alpha(t))$ gives that $m''(a,b) = m''(b,a)$.

c.) If H is not connected, there is no group structure which makes p a group homo and f a neutral elem. in general.



<u>Example</u> $H = S^1 \amalg \mathbb{R}$ $\quad G = S^1 \ni e = f$

$p = (\text{covering } S^1 \xrightarrow{id} S^1) \amalg (\text{covering } \mathbb{R} \xrightarrow{t \mapsto \exp it} S^1)$

Assume by contradiction there is such a multiplication m'' on H.

continued on p.3

1.) c.) continued.) Choose $x \in \mathbb{R}$

Define $\gamma: S^1 \to H$, $s \mapsto m''(s,x)$ cont. since m'' is cont.

$\gamma(\overset{\in S^1}{f}) = m''(f,x) = x \in \mathbb{R}$ } $\Rightarrow$ We can restrict the codomain of γ to get a $\gamma': S^1 \to \mathbb{R}$

S^1 connected $\Rightarrow im(\gamma)$ connected

γ is injective since H is a group $\Rightarrow \gamma'$ is injective

But there is no cont. injection $S^1 \to \mathbb{R}$ $\lightning$ contradiction.

d.) p is a covering $\rightsquigarrow$ get disjoint neighbourhoods V_x of $x \in p^{-1}(e)$

$\Rightarrow$ every point $x \in p^{-1}(e)$ is open in the subspace top. of $ker(p) = p^{-1}(e)$

$\Rightarrow ker(p)$ is discrete

$ker(p)$ is normal as the kernel of a group homo.

e.) $Aut_G(H) = \{s: H \to H \text{ homeo} \mid p = p \circ s\}$

<u>ϑ well def.</u> For $k \in ker\, p$, $h \in H$ $\quad p(k \cdot h) = \underbrace{p(k)}_{=e} \cdot p(h) = p(h)$

$h \mapsto kh$ is cont. since m'' is cont.

$h \mapsto kh$ is a homeo. since it has inverse $h \mapsto k^{-1}h$

<u>ϑ group homo</u>: $(h \mapsto k \cdot l \cdot h) = (h' \mapsto k \cdot h') \circ (h \mapsto l \cdot h)$

<u>ϑ bijective</u>: For $s \in Aut_G(H)$ (i.e. $s: H \to H$ homeo with $p = p \circ s$)

Define $k := s(f) \in ker(p)$ bec. $p(s(f)) = p(f) = e$

Choose a path α from f to a

$p(s(\alpha(t))) = (p \circ s)(\alpha(t)) = p(\alpha(t)) = \overbrace{p(k)}^{e} \cdot p(\alpha(t)) = p(k \cdot \alpha(t))$

and $s(\alpha(0)) = s(f) = k = k \cdot f = k \cdot \alpha(0)$

$\xRightarrow{p \text{ covering}} s(a) = k \cdot a \xrightarrow{a \text{ arbitrary}} s: a \mapsto k \cdot a \xRightarrow{s \text{ arbitrary}} \vartheta$ surj.

ϑ is injective bec. $(h \mapsto k \cdot h) = id$ implies $\underset{k=}{k \cdot f = f}$

f.) Since $\tilde{G} \xrightarrow{p} G$ is surjective we have $1 \to ker\,p \to \tilde{G} \xrightarrow{p} G \to 1$

and it suffices to show that $\pi_1(G,e)^{op} \cong ker\, p$

$\pi_1(G,e)^{op} \cong \pi_1(G,e) \longrightarrow ker(p)$, $[\alpha] \mapsto$ (endpoint $\tilde{\alpha}(1)$ of lift $\tilde{\alpha}$ of α with startingpoint $\tilde{\alpha}(0) = f$.)

<u>well def.</u>: $p \circ \tilde{\alpha} = \alpha \Rightarrow p(\tilde{\alpha}(1)) = \alpha(1) = e$

<u>group homo</u>: $\alpha * \beta \sim_{htp} \alpha \cdot \beta \Longrightarrow \widetilde{\alpha * \beta} \sim_{htp} \tilde{\alpha} \cdot \tilde{\beta} \Rightarrow \widetilde{\alpha * \beta}(1) = \widetilde{\alpha \cdot \beta}(1) = \tilde{\alpha}(1) \cdot \tilde{\beta}(1)$

continued on p. 4

1.) f.) continued.)

<u>Injective</u>: $\tilde{\alpha}(1) = f = \tilde{\alpha}(0) \xRightarrow{\tilde{G} \text{ universal}} \tilde{\alpha} \underset{\text{htp}}{\sim} \text{const path} \Rightarrow \alpha = p \circ \tilde{\alpha}$ const. path

<u>surjective</u> For $h \in \ker(p)$: $p(h) = e$ take a path $\mathcal{K}$ from f to h.

$[p \circ \mathcal{K}] \in \pi_1(G, e)$; $p \circ h$ lifts to $\mathcal{K}$ with endpoint h.

qed.

Since we only used the universality of the covering $\tilde{G}$ for the injectivity we get for a general covering $H \xrightarrow{p} G$ that there is a right exact sequence $\pi_1(G, e) \xrightarrow{\partial} H \xrightarrow{p} G \to 1$.

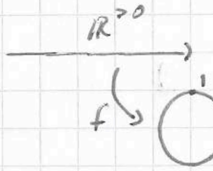
2 We prove this by contradiction, i.e. assume $K \not\subset Z(H)$. [4]

Thus there exists $k \in K \setminus Z(H)$ and a $\tilde{g} \in H$ st $[k:\tilde{g}] \neq e$.

We notice that the map $g \mapsto [k:g]$ is continuous. We write $G = \{g \in H \mid [k:g] = e\} \cup \{g \in H \mid [k:g] \neq e\}$. Both sets are disjoint. The first is just the preimage of e und $[k:-]$ and is thus open. (since K is discrete, we know that $\{e\} = K \cap U$ for U open in H. Thus $\{g \in H \mid [k:g] = e\} = [k:-]^{-1}(K \cap U) = [k:-]^{-1}(K) \cap [k:-]^{-1}(U)$, where by normality $[k:-]^{-1}(K) = H$ and $[k:-]^{-1}(U)$ open.) We now show that the second set is open. The second set is $[k:-]^{-1}(K \setminus \{e\}) = \bigcup_{k' \in K \setminus \{e\}} [k:-]^{-1}(k')$ which is the union of open sets by the previous argument.

Furthermore, both sets are non-empty: The first contains k itself and the second contains $\tilde{g}$. Thus we have decomposed H into two disjoint, non-empty open sets, which contradicts the connectedness of H. Hence $K \subset Z(H)$. But since $Z(H)$ commutes, $K \triangleleft Z(H)$.

3) a.) Consider $f: \mathbb{R}^{>0} \to S^1$, $s \mapsto e^{i2\pi s}$



<u>local homeo:</u> Let $x \in \mathbb{R}^{>0}$

$x \in \mathbb{R}^{>0} \cap (x-\frac{1}{2}, x+\frac{1}{2}) =: V_x$ is an open n.b.h. of x.

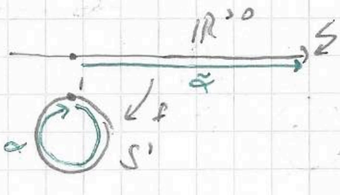
$f|_{V_x}$ is a restriction of the homeo $(x-\frac{1}{2}, x+\frac{1}{2}) \xrightarrow{\sim} S^1 \setminus f(x+\frac{1}{2})$ $\;s \mapsto e^{i2\pi s}$

and therefore a homeo onto its image, which is open.

<u>not a covering</u> because the path $\alpha: s \mapsto e^{i2\pi(-s)}$

cannot be lifted to a path $\tilde{\alpha}$ in $\mathbb{R}^{>0}$

with starting point $\tilde{\alpha}(0) = 1$



b.) assuming f is surjective:

Choose for every $x \in f^{-1}(\{y\})$ an $x \in U_x \subset X$ and a $y \in V_x \subset Y$

such that $U_x \xrightarrow{\sim} V_x$ under f.

For $x \neq x'$ ($x' \in f^{-1}(\{y\})$) $x' \notin U_x$ since otherwise $f(x) = f(x')$, $x \neq x'$ on U_x contradicts $U_x \xrightarrow[\sim]{f|_{U_x}} V_x$

$\Rightarrow \bigcup_{x \in f^{-1}\{y\}} U_x \cup f^{-1}(Y \setminus \{y\})$ is an open cover of X such that

every subcover must at least contain all U_x for $x \in f^{-1}(\{y\})$

$\xRightarrow{X \text{ compact}}$ $f^{-1}\{y\}$ is finite.

X Hausdorff $\xrightarrow{f^{-1}\{y\} \text{ finite}}$ w.l.o.g. $U_x \ni x$ disjoint. $\left(\begin{array}{l}\text{otherwise restrict to} \\ \text{smaller neighbourhoods}\end{array}\right.$ $\begin{array}{ccc} U_x & \xrightarrow{\sim} & V_x \\ \cup & & \cup \\ U'_x & \xrightarrow{\sim} & V'_x \end{array}\Big)$

$V := \bigcap_{x \in f^{-1}\{y\}} V_x$ $\quad$ w.l.o.g. $U_x = f^{-1}(V) \cap U_x$

$\Rightarrow V_x = V$

$\xRightarrow{x \text{ arbitrary}}$ f is a covering $\quad$ qed.