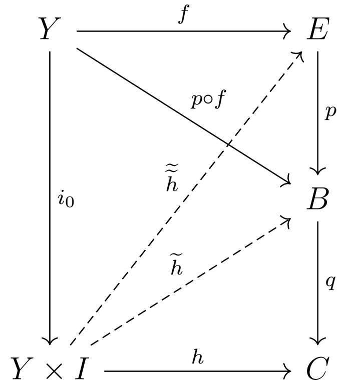
Exercise Sheet 5

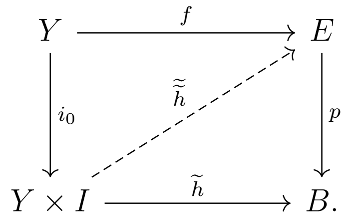
2. Let $p\colon E \to B$ and $q\colon B \to C$ be fibrations. [In May's book, fibrations are surjective.]

   Claim. *The composition $q \circ p$ is a fibration.*

   Proof. $q \circ p$ is surjective. Consider a test diagram for the CHP with Y, f, h given.



   Since q is a fibration, h has a lift $\widetilde{h}\colon Y \times I \to B$ with $q \circ \widetilde{h} = h$ and $p \circ f = \widetilde{h} \circ i_0$. Note that the last condition implies that the following diagram commutes:



   Since p is a fibration, $\widetilde{h}$ has a lift $\widetilde{\widetilde{h}}\colon Y \times I \to B$ with $p \circ \widetilde{\widetilde{h}} = \widetilde{h}$ and $\widetilde{\widetilde{h}} \circ i_0 = f$. In particular,

$$q \circ p \circ \widetilde{\widetilde{h}} = q \circ \widetilde{h} = h.$$

$\square$

3. Consider the map

$$p\colon \mathrm{GL}_2(\mathbb{R}) \longrightarrow \mathbb{R}^2 \setminus \{0\}$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (a, b)$$

   (a) p is evidently surjective, since for any nonzero $(a, b) \in \mathbb{R}^2$, the matrix

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

   is invertible (it has determinant $a^2 + b^2 > 0$). In order to show that p satisfies the CHP, we show that p is a fiber bundle.

Identify $\mathbb{R}^2 \setminus \{0\}$ with $\mathbb{C}^\times$ and define $F = \mathbb{C} \setminus \mathbb{R}$. Consider the map

$$\begin{aligned} \varphi\colon \mathrm{GL}_2(\mathbb{R}) &\longrightarrow \mathbb{C}^\times \times F \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto (a+bi, (c+di)/(a+bi)) . \end{aligned}$$

This is well-defined ($a + bi \neq 0$; $(c+di)/(a+bi) \notin \mathbb{R}$ since (a,b) and (c,d) are linearily independent over $\mathbb{R}$) and continuous. Its inverse is given by

$$(u,v) \longmapsto \begin{pmatrix} \operatorname{Re} u & \operatorname{Im} u \\ \operatorname{Re} uv & \operatorname{Im} uv \end{pmatrix} .$$

Evidently the diagram

$$\begin{array}{ccc} \mathrm{GL}_2(\mathbb{R}) & \xrightarrow{\varphi} & \mathbb{C}^\times \times F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & \mathbb{C}^\times & \end{array}$$

commutes, so φ is a (global) trivialization. It follows from the following lemma (which is implied by the fact that fibrations are stable under pullback) that p is a fibration.

Lemma. *Let X and Y be topological spaces. The canonical projection map $\mathrm{pr}_1\colon X \times Y \to X$ is a fibration.*

Proof. Consider a test diagram for the covering homotopy property with Z, f, h given.

$$\begin{array}{ccc} Z & \xrightarrow{\quad f \quad} & X \times Y \\ {\scriptstyle i_0}\big\downarrow & \overset{\widetilde{h}}{\nearrow} & \big\downarrow{\scriptstyle \mathrm{pr}_1} \\ Z \times I & \xrightarrow{\quad h \quad} & X \end{array}$$

The map

$$\begin{aligned} \widetilde{h}\colon Z \times I &\longrightarrow X \times Y \\ (z,t) &\longmapsto (h(z,t), \mathrm{pr}_2(f(z))) \end{aligned}$$

works. □

(b) In (a) we saw that p is trivial; in particular it is locally trivial.

(c) Since p is a trivial fibration, the action is trivial (loops lift to loops).