1. [Compression criterion.] Let X be a topological space, $A \subset X$ and $x_0 \in A$.

Claim. *Let $n \geq 0$ be an integer. A map $f\colon (I^n, \partial I^n, J^n) \to (X, A, x_0)$ represents the trivial class in $\pi_n(X, A, x_0)$ iff it is homotopic rel ∂I^n to such a map of triples with image in A.*

Proof. Suppose first that f is homotopic rel ∂I^n to such a map g. In particular, f and g represent the same class in $\pi_n(X, A, x_0)$. Define $\varphi\colon I^n \times I \to I^n$ by the formula $\varphi(x_1, \ldots, x_n, t) = (x_1, \ldots, x_{n-1}, t + (1-t)x_n)$ (we move all points away from $I^{n-1} \subset I^n$). Then $h\colon I^n \times I \to X$ with $h(x,t) = g(\varphi(x,t))$ is a homotopy from g to the constant map c_{x_0}, so $[f] = [g] = 0 \in \pi_n(X, A, x_0)$.

Conversely, suppose that $h\colon I^n \times I \to X$ is a homotopy from f to the constant map. Define $\varphi\colon I \times I \to I \times I$ by

$$\varphi(x,t) = \begin{cases} (0, (1+t)x)\,, & 0 \leq x \leq t/(1+t) \\ ((1+t)(x - t/(1+t)), t)\,, & t/(1+t) \leq x \leq 1. \end{cases}$$

Define $\widetilde{h}\colon I^n \times I \to X$ by $\widetilde{h}(x_1, \ldots, x_n, t) = h(x_1, \ldots, x_{n-1}, \varphi(x_n, t))$. This is a homotopy rel ∂I^n from f to a map with image in A. □

3. Let (X, A) be a pair of spaces with A path-connected. Recall that $\pi_1(X, A, x_0)$ consists of all maps $\alpha\colon (I, \{0,1\}, 1) \to (X, A, x_0)$ modulo homotopies of maps of triples. Let H be the subgroup of the ordinary fundamental group $\pi_1(X, x_0)$ consisting of all classes $[\alpha]$, where α is a loop at x_0 in A.

There is a canonical map $i_*\colon \pi_1(X, x_0) = \pi_1(X, x_0, x_0) \to \pi_1(X, A, x_0)$, induced by the inclusion. We show that this factors over the quotient set $\pi_1(X, x_0)/H$. Given any $[\alpha] \in H$, we have $i_*[\alpha] = 0$ by the compression criterion. Thus we obtain an induced map $\varphi\colon \pi_1(X, x_0)/H \to \pi_1(X, A, X_0)$.

φ is injective: Suppose that α, β are loops at x_0 such that $\varphi([\alpha]H) = \varphi([\beta]H)$. This means that $[\alpha^{-1} \cdot \beta] = [c_{x_0}]$ in $\pi_1(X, A, x_0)$, so by the compression criterion the loop $\alpha^{-1} \cdot \beta$ is homotopic rel $\{0,1\}$ via a homotopy h to a map $\gamma\colon I \to X$ with image in A. Since h is a homotopy rel $\{0,1\}$, it is a homotopy of paths and γ is a loop at x_0 in A. In particular, $[\alpha^{-1} \cdot \beta] = [\gamma] \in H$.

φ is surjective: Consider a map $\alpha\colon (I, \{0,1\}, 1) \to (X, A, x_0)$, which might not be a loop. Since A is path-connected, we can choose a path γ from x_0 to $\alpha(0)$ in A. Then $\alpha \cdot \gamma$ is a loop at x_0 and $[\alpha] = [\alpha \cdot \gamma]$ lies in the image of φ.

2.) Let $2 \le k \le n$ and let $+'$ be the sum using coordinate k.

$$\big((f+g) +' (h+k)\big)(v_1, \dots v_n) = \begin{cases} (f+g)(v_1, \dots v_{k-1}, 2 v_k, v_{k+1}, \dots v_n) & v_k \le \frac{1}{2} \\ (h+k)(v_1, \dots v_{k-1}, 2v_k - 1, v_{k+1}, \dots v_n) & v_k \ge \frac{1}{2} \end{cases}$$

$$= \begin{cases} f(2v_1, v_2, \dots v_{k-1}, 2 v_k, v_{k+1}, \dots v_n) = (f+'h)(2v_1, v_2, \dots v_n) & v_1 \le \frac{1}{2} \quad v_k \le \frac{1}{2} \\ g(2v_1 - 1, v_2, \dots v_{k-1}, 2v_k, v_{k+1}, \dots v_n) = (g+'k)(2v_1 - 1, v_2, \dots v_n) & v_1 \ge \frac{1}{2} \quad v_k \le \frac{1}{2} \\ h(2v_1, v_2, \dots v_{k-1}, 2v_k - 1, v_{k+1}, \dots v_n) = (f+'h)(2v_1, v_2, \dots v_n) & v_1 \le \frac{1}{2} \quad v_k \ge \frac{1}{2} \\ k(2v_1 - 1, v_2, \dots v_{k-1}, 2v_k - 1, v_{k+1}, \dots v_n) = (g+'k)(2v_1 - 1, v_2, \dots v_n) & v_1 \ge \frac{1}{2} \quad v_k \ge \frac{1}{2} \end{cases}$$

$$= \left\{ \begin{matrix} (f+'h)(2v_1, v_2, \dots v_n) & v_1 \le \frac{1}{2} \\ (g+'k)(2v_1 - 1, v_2, \dots v_n) & v_1 \ge \frac{1}{2} \end{matrix} \right\} = \big((f+'h) + (g+'k)\big)(v_1, \dots v_n)$$

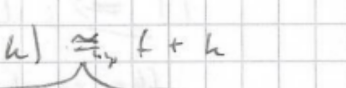
2.) continued.)

$f \simeq_{htp} f + c_{x_0} \wedge k \simeq_{htp} c_{x_0} + k$

$\Rightarrow f +' h \simeq_{htp} (f + c_{x_0}) +' (c_{x_0} + k) \overset{\text{above}}{=\!=} (f +' c_{x_0}) + (c_{x_0} +' h) \simeq_{htp} f + h$

$f +' c_{x_0} \simeq_{htp} f \wedge c_{x_0} +' h \simeq_{htp} h$

Analogous: $g +' h \simeq_{htp} (c_{x_0} + g) +' (h + c_{x_0}) \overset{\text{above}}{=} (c_{x_0} +' h) + (g +' c_{x_0}) \simeq_{htp} h + g$



The sum $f + g$ of maps $f, g : (I^n, \partial I^n, J^n) \to (X, A, x_0)$ is defined using the second coordinate

We can instead define $f +' g$ using coordinate h for $h \geq 2$

analogous to the non-relative case we see that

$(f + g) +' (h + k) \simeq (f +' h) + (g +' k)$ for $f, g, h, k : (I^n, \partial I^n, J^n) \to (X, A, x_0)$

and we can deduce analogously that $g +' h$ is homotop rel. ∂I^n

to $h + g$.

4.) Consider the maps $j : (X, x_0, x_0) \hookrightarrow (X, A, x_0)$ and $i : (A, x_0) \hookrightarrow (X, x_0)$

and $\partial : \pi_1(X, A, x_0) \to \pi_0(A, x_0)$, $[\alpha : (I^1, \partial I^1, J^1) \to (X, A, x_0)] \mapsto [\alpha|_{I^0}] = [\alpha(1)]$

$I^1 = [0,1]$ $\quad \partial I^1 = \{0,1\}$ $\quad J^1 = \{0\}$ $\quad I^0 = \{1\}$

$\pi_1(X, x_0) \xrightarrow{j_*} \pi_1(X, A, x_0) \xrightarrow{\partial} \pi_0(A, x_0) \xrightarrow{i_*} \pi_0(X, x_0)$

exactness at $\pi_1(X, A, x_0)$:

$\partial \circ j_* = 0$: Consider $\alpha : (I^1, \partial I^1, J^1) \to (X, x_0, x_0)$

$$\partial \circ j_*(\alpha) = j_*(\alpha)|_{I^0} \overset{I^0 \subset \partial I^1}{=\!=\!=} c_{x_0}$$

Conversely consider $\alpha : (I^1, \partial I^1, J^1) \to (X, A, x_0)$ with $\partial(\alpha) = c_{x_0} \Rightarrow \alpha|_{I^0} = c_{x_0}$

With $\alpha|_{J^1} = c_{x_0}$ and $\partial I^1 = J^1 \cup I^0$ it follows that $\alpha|_{\partial I^1} = c_{x_0}$

$\rightsquigarrow \alpha$ induces a map $\tilde{\alpha} : (I^1, \partial I^1, J^1) \to (X, x_0, x_0)$

i.e. $j_*(\tilde{\alpha}) = \alpha$

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(3)

$i_* \circ \partial = 0$ consider $\alpha : (I^1, \partial I^1, J^1) \to (X, A, x_0)$

$(i_* \circ \partial)\alpha = [\underbrace{\alpha(1)}_{\in X}]$

The path α connects $\alpha(1)$ and $\alpha(0) = x_0$. $\Rightarrow$ same path conn. comp.

$\Rightarrow [\alpha(1)] = [x_0]$

Conversely let $[a] \in \pi_0(A, x_0)$ with $[x_0]_{\pi_1(X, x_0)} = i_*([a]_{\pi_1(A, x_0)}) = [a]_{\pi_1(X, x_0)}$

$\Rightarrow x_0, a$ in same path conn. comp $\rightsquigarrow$ get a path α from x_0 to a

$\Rightarrow \partial[\alpha] = [a]$

5. Let $n \geq 1$ be an integer. We embed $i\colon S^{n-1} \to S^n$ as the equator and fix some basepoint $x_0 \in S^{n-1}$. Consider the long exact sequence for the relative homotopy groups:

$$\cdots \longrightarrow \pi_n(S^{n-1}, x_0) \xrightarrow{\pi_n(i)} \pi_n(S^n, x_0) \xrightarrow{\varphi} \pi_n(S^n, S^{n-1}, x_0)$$

$$\downarrow$$

$$\cdots \longleftarrow \pi_{n-1}(S^n, x_0) \longleftarrow \pi_{n-1}(S^{n-1}, x_0)$$

$$\cdots \longrightarrow \pi_k(S^n, x_0) \longrightarrow \pi_k(S^n, S^{n-1}, x_0) \longrightarrow \pi_{k-1}(S^{n-1}, x_0) \longrightarrow \cdots$$

$$\cdots \longrightarrow \pi_1(S^n, x_0) \longrightarrow \pi_1(S^n, S^{n-1}, x_0).$$

We know that $\pi_k(S^n) = 0$ for $k < n$ and that $\pi_n(S^n) = H_n(S^n, \mathbb{Z}) = \mathbb{Z}$ (Hurewicz). In particular, the first fact implies that $\pi_k(S^n, S^{n-1}, x_0) = 0$ for $1 < k < n$. For $k = 1$, we have a bijection $\pi_1(S^n, S^{n-1}, x_0) \cong \pi_1(S^n, x_0)/H$, where H is defined as in Exercise 3. If $n > 1$, then $\pi_1(S^n, x_0) = 0$ and therefore $\pi_1(S^n, S^{n-1}, x_0) = 0$. If $n = 1$, then H is trivial (the only loop at x_0 in S^0 is the constant loop), so $\pi_1(S^1, S^0, x_0) = \pi_1(S^1, x_0) = \mathbb{Z}$.

It remains to compute $\pi_n(S^n, S^{n-1}, x_0)$. If $\alpha\colon S^n \to S^{n-1}$ represents a class in $\pi_n(S^{n-1}, x_0)$, then $\pi_n(i)[\alpha]$ is represented by the composition

$$S^n \xrightarrow{\alpha} S^{n-1} \xhookrightarrow{i} S^n.$$

The image of this point lies in the contractible space $S^n \backslash \{\mathrm{pt}\}$, so it is nullhomotopic. In particular, $\pi_n(i)$ is the trivial map. By exactness of the sequence, this implies that φ is injective. Thus we have the split short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_n(S^n, S^{n-1}, x_0) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

so $\pi_n(S^n, S^{n-1}, x_0) = \mathbb{Z} \oplus \mathbb{Z}$ by the splitting lemma.

$$\pi_{n-1}(S^{n-1}) \quad \cong$$

Thus we have $0 \to \mathbb{Z} \to \pi_n(S^n, S^{n-1}) \to \mathbb{Z} \to 0$. This sequence splits, since any map $(I^{n-1}, \partial I^{n-1}, x_0) \to (I^{n-1}, \partial I^{n-1}, x)$ can be extended to a map $(I^n, I^{n-1} = \partial I^n, x_0) \to (I^n, \partial I^n, x_0)$ which on ∂I^{n-1} is just the given map.

Hence we have $\pi_n(S^n, S^{n-1}) = \mathbb{Z}^2$

6a) We consider $X = S^3$ and A two circles glued together (in ∞) and get the following sequence

$$\pi_2(A) \to \pi_2(X) \to \pi_2(X, A) \to \pi_1(A) \to \pi_1(X)$$
$$\pi_2(X) = 0, \quad \pi_1(A) = \mathbb{Z} * \mathbb{Z}, \quad \pi_1(X) = 0$$

ie $\pi_2(X, A) = \mathbb{Z} * \mathbb{Z}$.

7. We first notice that for any n, $S^{n-1} \hookrightarrow D^n$ is a cofibration, and since cofibration are stable under product and pushout, we get

$$\begin{array}{ccc} I^n \times S^{n-1} & \xrightarrow{\varphi} & X^{n-1} \\ \downarrow & \lrcorner & \downarrow \\ I^n \times D^n & \longrightarrow & X^n \end{array}$$

for φ the map given by the cell-structure of X that $X^{n-1} \hookrightarrow X^n$ is a cofibration

$\left(X^n \text{ is } \left(X_{n-1} \cup (I^n \times D^n)\right) / (i,s) \sim \varphi(i,s) \ \forall s \in S^{n-1}, i \in I^n \right)$

Furthermore, since $A^0 \hookrightarrow X^0$ is a cofibration as well (they are both discrete sets), we can prove that $A^n \hookrightarrow X^n$ is a cofibration:

$$\begin{array}{ccc} A^i & \overset{\text{cofib}}{\hookrightarrow} & X^i \\ \downarrow & \lrcorner & \downarrow \\ A^{i+1} & \hookrightarrow & X_i \end{array}$$

since the pushout along $A^i \hookrightarrow X_i$ and $A^i \xrightarrow{c} A^{i+1}$ is just X^i.

Now we can show that for any i, $A^i \to X$ is a cofibration.

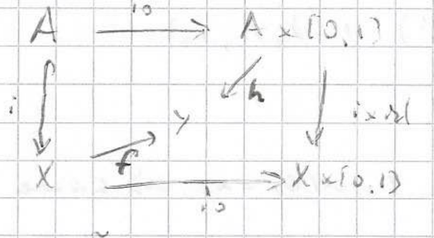
$$\begin{array}{ccc} A^i & \longrightarrow & Y^{[0,1]} \\ \downarrow & \nearrow & \downarrow \\ X^i & \longrightarrow & Y \\ \downarrow & \nearrow & \\ X^{i+1} & & \end{array} \qquad , \qquad \begin{array}{ccc} X^i & \longrightarrow & Y \\ \downarrow & \nearrow & \\ X & & \end{array}$$

Given any map $A^i \to Y^{[0,1]}$ and $X \to Y$ which induces maps $X_i \to Y$, for any j, one can find a map from X^j to Y to solve the HEP, iteratively using the fact that $X^i \hookrightarrow X^{i+1}$ is a cofibration, st.

$$\begin{array}{ccc} X^i & \longrightarrow & Y^{[0,1]} \\ \downarrow & \nearrow & \\ X^{i+1} & & \end{array}$$

. This gives us a well-defined map $X \to Y^{[0,1]}$.

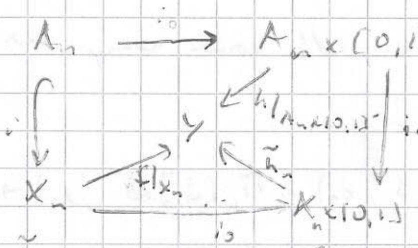
7.) Let



be a test diag.

To construct $\tilde{h} : X \times [0,1] \to Y$, we construct inductively a system $\tilde{h}_n : X_n \times [0,1] \to Y$ for $n \in \mathbb{Z}^{\geq 0}$

with commutative



and $\tilde{h}_n|_{X_{n-1}} = \tilde{h}_{n-1}$.

$\underline{n=0}$: $\tilde{h}_0 : X_0 \times [0,1] \to Y$, $(x,t) \mapsto \begin{cases} h(x,t) & x \in A \\ f(x) & x \notin A \end{cases}$

$\underline{n \geq 1}$: Consider a n-cell $i \in I_n$

<u>case $i \in J_n$</u> (Cell is contained in A)

Then h determines $\tilde{h}$ on this cell.

(β_n is the homeo $J_n \times D^n \amalg A_{n-1}/\sim \to A_n$ of the CW-structure)

Set $\tilde{h}_{n,i} = h \circ (A_n \times [0,1] \hookrightarrow A \times [0,1]) \circ (\beta_n \circ quot|_{\{i\} \times D^n} \times id_{[0,1]}) : D^n \times [0,1] \to Y$

<u>case $i \notin J_n$</u> (Cell is not contained in A)

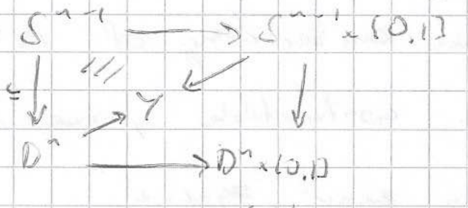
Then the previous induction step determines $\tilde{h}$ on the boundary of the cell:

The maps (φ is the map $I_n \times S^{n-1} \to X_n$ of the CW-structure)

$S^{n-1} \times [0,1] \xrightarrow{(\varphi|_{\{i\} \times S^{n-1}}) \times id_{[0,1]}} X_{n-1} \times [0,1] \xrightarrow{\tilde{h}_{n-1}} Y$

and $D^n \xrightarrow{\alpha_n \circ quot|_{\{i\} \times D^n}} X_n \hookrightarrow X \xrightarrow{f} Y$ where α_n is the homeo $I_n \times D^n \amalg X_{n-1}/\sim \to X_n$ of the CW structure

form a com. diag.



(because $I_n \times S^{n-1} \to X_{n-1}$, $I_n \times D^n \xrightarrow{\alpha_n \circ quot} X_n$ commutes)

Since $S^{n-1} \hookrightarrow D^n$ is a cofibration (similar proof to sheet 4 ex 5.c.)

we get a factor $\tilde{h}_{n,i} : D^n \times [0,1] \to Y$

These homotopies $\tilde{h}_{n,i}$ for $i \in I_n$ together with $\tilde{h}_{n-1} : X_{n-1} \times [0,1] \to Y$

give a homotopy $(I_n \times D^n \amalg X_{n-1}) \times [0,1] \to Y$

7.) continued.)

This homotopy factors through $I_n \times D^n \sqcup X_{n-1} \to I_n \times D^n \sqcup X_{n-1} \big/ {\scriptstyle (i,s)\sim\varphi(i,s),\ (i,s)\in I_n\times S^{n-1}} \xrightarrow{\alpha_n} X$

because: for $(i,s)\in I_n\times S^{n-1}$, $t\in[0,1]$

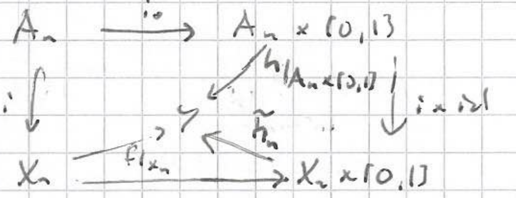
<u>case</u> $i\in J_n$: $\tilde h_n((i,s),t)=\tilde h_{n,i}(s,t)\overset{def.}{=}h(\beta_n(\underbrace{quot(i,s)}_{=\varphi(i,s)}),t)\overset{\alpha_{n-1}|_{A_{n-1}}=\beta_n|_{A_{n-1}}}{=\!=\!=}h(\alpha_{n-1}(\varphi(i,s)),t)$
$\overset{\text{previous ind. step}}{=\!=\!=}\tilde h_{n-1}(\varphi(i,s),t)$

<u>case</u> $i\notin J_n$: $\tilde h_n((i,s),t)=\tilde h_{n,i}(s,t)\underset{\text{constr. of }\tilde h_{n,i}}{\overset{(s,t)\in S^{n-1}\times[0,1]}{=\!=\!=}}\tilde h_{n-1}(\varphi(i,s),t)$

Therefore we get a homotopy

$\tilde h_n : X_n\times[0,1]\to Y$

From the constr. of $\tilde h_{n,i}$ it follows that



commutes

and that $\tilde h_n|_{X_{n-1}}=\tilde h_{n-1}$

Then the homotopy $\tilde h : X\times[0,1]\to Y$, defined as

$\tilde h(x,t)=\tilde h_n(x,t)$ for $n\geq 0$ with $x\in X_n$

is well def and factors the test diagram. qed