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Exercise Sheet 1

1. A *topological group* G is a group, together with a topology such that the multiplication map $G \times G \longrightarrow G$ and the inversion map $G \longrightarrow G$ are continuous.

a) Prove that if H is an open subgroup of G, then H is closed.

b) Is the converse true?

c) Prove that G is Hausdorff if and only if $\{1_G\}$ is closed.

2. Let $(I, \leq)$ be a partially ordered set which is *directed*, meaning that for each $\alpha, \beta \in I$ there exists $\gamma \in I$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$. A *filtered inverse system of groups indexed on* I is the data of groups G_α for $\alpha \in I$ and group morphisms $\phi_{\alpha\beta} : G_\beta \longrightarrow G_\alpha$ for each $\alpha \leq \beta$, such that $\phi_{\alpha\alpha} = \mathrm{id}_{G_\alpha}$ and $\phi_{\alpha\gamma} = \phi_{\alpha\beta}\phi_{\beta\gamma}$ for all $\alpha \leq \beta \leq \gamma$.

a) We define the limit of the filtered inverse system $(G_\alpha, \phi_{\alpha\beta})$ as

$$\lim_{\alpha \in I} G_\alpha := \{(g_\alpha)_{\alpha \in I} : \forall \alpha \leq \beta, \phi_{\alpha\beta}(g_\beta) = g_\alpha\} \leq \prod_{\alpha \in I} G_\alpha.$$

What is the universal property of $\lim_{\alpha \in I} G_\alpha$?

b) Suppose that the G_α are Hausdorff topological groups and endow $\prod_{\alpha \in I} G_\alpha$ with the product topology. Prove: $\lim_{\alpha \in I} G_\alpha$ is a closed subgroup of $\prod_{\alpha \in I} G_\alpha$.

We define a *profinite group* to be a limit of a filtered inverse system of **finite** discrete groups, with topology induced by the inclusion in the product. Given a group G, we define the *profinite completion* of G as

$$\hat{G} := \lim_{\substack{N \trianglelefteq G \\ [G:N] < \infty}} G/N.$$

c) Define the canonical map $\iota : G \longrightarrow \hat{G}$. Prove that it is injective if and only if G is *residually finite*, that is, for each element of G there is a group homomorphism h from G to a finite group such that $h(g) \neq 1$.

c') (Prompted by a question by João Manuel Pereira) Recall that a group G is called divisible if for each $g \in G$ and each integer $n \geq 1$, there exists an element $h \in G$ such that $h^n = g$. Show that there are no non-trivial group morphisms from a divisible group to a finite group. Deduce that a non-trivial divisible group cannot be residually finite. Use this to give concrete examples of groups which are not residually finite.

Please turn over!

d) We will deal during the course with the following profinite groups: $\hat{\mathbb{Z}}$, $\hat{\mathbb{Z}}^\times$ and $\mathbb{Z}_p$, with p a prime number. The profinite group $\mathbb{Z}_p$ is defined as the limit of the inverse system $(\mathbb{Z}/p^k\mathbb{Z})_{k\in\mathbb{Z}_{\geq 0}}$ with canonical projections as maps $\mathbb{Z}/p^k\mathbb{Z} \longrightarrow \mathbb{Z}/p^h\mathbb{Z}$ for $h \leq k$. How would you define $\hat{\mathbb{Z}}^\times$?

e) Prove that $\hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$ as topological groups, where p ranges over the prime numbers.

3. Let L/K be a finite field extension. The following statements are all equivalent definitions of the extension L/K being *Galois*:

- L/K is normal and separable;
- L is the splitting field of a separable polynomial $f \in K[X]$;
- $|\mathrm{Aut}_K(L)| = [L : K]$;
- $K = L^{\mathrm{Aut}(L/K)}$.

a) Prove that the four definitions above are equivalent.

b) State the Galois correspondence

$$\{L/M/K \text{ intermediate fields}\} \leftrightarrow \{H \leq \mathrm{Gal}(L/K)\}.$$

c) Let $\xi_n = e^{\frac{2\pi i}{n}} \in \mathbb{C}$. The cyclotomic field has a subfield $\mathbb{Q}(\xi_n + \xi_n^{-1})$. Identify the corresponding subgroup of $\mathrm{Gal}(\mathbb{Q}(\xi_n)/\mathbb{Q})$.

d) Compute all intermediate fields of $\mathbb{Q}(\xi_7)/\mathbb{Q}$ and the corresponding subgroups of $\mathrm{Gal}(\mathbb{Q}(\xi_7)/\mathbb{Q})$. Compare the result to Table 5.5 from the class (available on the webpage as well).

4. Prove that the only subgroups of finite index of $\mathbb{R}^\times$ are $\mathbb{R}_{>0}$ and $\mathbb{R}^\times$ itself. Explain how you would define the Artin map

$$\rho_{\mathbb{R}} : \mathbb{R}^\times \longrightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{R}).$$

5. Prove: all local fields are complete.

6. **a)** Prove that $|\cdot|_p$ is an absolute value on $\mathbb{Q}$.

b) Prove that the map $|\cdot|$ on $k((t))$ defined in class is indeed an absolute value.

c) Show that both absolute values above are non-archimedean.