Exercise Sheet 10

Throughout, G denotes a finite group and, if p is a prime number, G_p a p-Sylow subgroup of G. Recall that a G-module A is said to be *cohomologically trivial* if $H^n(H, A) = 0$ for all integers $n > 0$ and all subgroups H of G.

1. The goal of this exercise is to prove the following:

Theorem 1. *Let G be a finite group and A a G-module.*

(i) *Suppose that, for each prime number p, there exists an integer n_p such that*

$$\widehat{H}^{n_p}(G_p, A) = \widehat{H}^{n_p+1}(G_p, A) = 0.$$

Then the G-module A is cohomologically trivial. Moreover, if A is a free abelian group, then A is a projective $\mathbb{Z}[G]$-module.

(ii) *Assume A is cohomologically trivial. Then $\widehat{H}^q(H, A) = 0$ for all $q \in \mathbb{Z}$ and all subgroups H of G. Moreover, A fits into an exact sequence*

$$0 \to R \to F \to A \to 0,$$

where F is a free $\mathbb{Z}[G]$-module and R is a projective $\mathbb{Z}[G]$-module.

a) We write A as a quotient F/R, where F is a free $\mathbb{Z}[G]$-module. Note that R is torsion-free. Using the short exact sequence

$$0 \to R \xrightarrow{\cdot p} R \to R/pR \to 0$$

and the assumptions of (i), show that

$$\widehat{H}^{n_p+1}(G_p, R/pR) = 0.$$

b) Deduce from Theorem 1 in Exercise sheet 9 that R/pR is an induced G_p-module and, in particular, cohomologically trivial.

c) We first assume that A is a free abelian group. Set $M = \mathrm{Hom}(A, R)$. Our first goal is to show that $H^1(G, M) = 0$. First observe that

$$M/pM \simeq \mathrm{Hom}(A, R/pR),$$

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hence M/pM is an induced G_p-module. From the long exact sequence of modified cohomology associated to the short exact sequence

$$0 \to M \xrightarrow{\cdot p} M \to M/pM \to 0$$

and the fact that the restriction map $H^1(G,M)\{p\} \to H^1(G_p, M)$ is injective, deduce that, for all prime numbers p, one has

$$H^1(G,M)\{p\} = 0.$$

Therefore, $H^1(G,M) = 0$.

d) Always under the assumption that A is a free abelian group, prove that there is an exact sequence of G-modules

$$0 \to M \to \mathrm{Hom}(A,F) \to \mathrm{Hom}(A,A) \to 0.$$

Deduce that there is a surjection $\mathrm{Hom}_G(A,F) \to \mathrm{Hom}_G(A,A)$. By considering any pre-image of the identity id_A, show that $F = A \oplus R$ *as G-modules.* This yields part (i) of the theorem whenever A os a free abelian group.

e) Now, for general A, deduce that A is cohomologically trivial by applying the previous arguments to the free abelian group R.

f) Finally, let A be a cohomologically trivial G-module. Observe that A satisfies the assumptions of part (i), for example for $n_p = 1$. Write $A = F/R$ with F a free $\mathbb{Z}[G]$-module and deduce part (ii) of the theorem.

2. Let A be a cohomologically trivial G-module. Prove that, for all torsion free G-modules D, the G-module $A \otimes D$ is cohomologically trivial as well.

(*Hint:* by Theorem 1, there is an exact sequence $0 \to R \to F \to A \to 0$ where F and R are free and projective $\mathbb{Z}[G]$-modules respectively).

3. The goal is to prove the following:

Proposition 1. *Let $f\colon A \to B$ be a morphism of G-modules. Assume that, for each prime number p, there exists an integer n_p such that the map*

$$f_*^q\colon \widehat{H}^q(G_p, A) \longrightarrow \widehat{H}^q(G_p, B)$$

is surjective for $q = n_p$, bijective for $q = n_p + 1$ and injective for $q = n_p + 2$. Then, for all torsion-free G-modules D and all subgroups H of G, the map

$$\widehat{H}^q(H, A \otimes D) \longrightarrow \widehat{H}^q(H, B \otimes D)$$

induced by $f \otimes 1$ is an isomorphism for all integers $q \in \mathbb{Z}$.

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a) We first assume that f is injective and let C denote its cokernel. Deduce from Theorem 1 that the G-module C is cohomologically trivial.

b) Using the assumption that D is torsion-free and Exercise 2, deduce the statement of the proposition when f is injective.

c) Given any f, consider the map $\theta\colon A \to B \oplus I_G(A)$ given by f and the natural injection $\iota\colon A \hookrightarrow I_G(A)$ which sends an element a to the function $g \mapsto g \cdot a$. Deduce the general statement from the injective case applied to θ.

4. In this exercise we show a proposition a special case of which will yield the Tate-Nakayama theorem:

Proposition 2. *Let A, B, C be three G-modules and $\phi\colon A \times B \to C$ a bilinear map compatible with the action of G. Fix an integer $q \in \mathbb{Z}$ and an element $a \in \widehat{H}^q(G, A)$. For each subgroup H of G and each torsion free G-module D, let*

$$f(n, H, D)\colon \quad \widehat{H}^n(H, B \otimes D) \longrightarrow \widehat{H}^{n+q}(H, C \otimes D)$$

be the morphism given by cup-product with $\mathrm{Res}_H(a)$ with respect to the bilinear map ϕ. Assume that, for all prime numbers p, there exists an integer $n_p \in \mathbb{Z}$ such that

- *$f(n_p, G_p, \mathbb{Z})$ is surjective,*
- *$f(n_p + 1, G_p, \mathbb{Z})$ is bijective,*
- *$f(n_p + 2, G_p, \mathbb{Z})$ is injective.*

Then the map $f(n, H, D)$ is bijective for all integers $n \in \mathbb{Z}$, all subgroups H of G and all torsion-free G-modules D.

a) Prove that the case $q = 0$ reduces to the statement of Proposition 1.

b) Deduce the general case via the shift trick and the fact that the cup-product commutes with coboundaries. More precisely, consider the injections

$$A \hookrightarrow \overline{A} := I_G(A), \quad C \hookrightarrow \overline{C} := I_G(C).$$

and set $A_1 = \overline{A}/A$ and $C_1 = \overline{C}/C$. The bilinear map $\overline{\phi}\colon \overline{A} \times B \to \overline{C}$ given by

$$\overline{\phi}(f, b) = (g \mapsto \phi(f(g), g \cdot b))$$

induces a bilinear map $\phi_1\colon A_1 \times B \to C_1$. Writing $a = \delta(a_1)$ with $a_1 \in \widehat{H}^{q-1}(G, A_1)$ and considering the cup-product by a_1

$$f_1(n, H, D)\colon \quad \widehat{H}^n(H, B \otimes D) \longrightarrow \widehat{H}^{n+q-1}(H, C_1 \otimes D),$$

deduce that, if the result holds for a_1, then it also holds for a. This yields the passage from $q-1$ to q.

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c) Do something similar to pass from q to $q-1$

5. Recall the statement of the Tate-Nakayama theorem:

Theorem 2 (Tate-Nakayama)**.** *Let A be a G-module and a an element of $H^2(G, A)$. Assume that the following two conditions hold for all primes p:*

- $H^1(G_p, A) = 0$,
- *The group $H^2(G_p, A)$ is cyclic of order $|G_p|$, generated by $\mathrm{Res}_{G_p}(a)$.*

Then, for all torsion-free G-modules D and all subgroups H of G, the map

$$\widehat{H}^n(H, D) \longrightarrow \widehat{H}^{n+2}(H, A \otimes D), \quad x \longmapsto x \cup \mathrm{Res}_H(a)$$

is an isomorphism for all integers n.

a) Show that the assumptions of Proposition 2 from the previous exercise hold for $B = \mathbb{Z}$, $C = A$, $\phi\colon A \times \mathbb{Z} \to A$ the obvious map, $q = 2$ and $n_p = -1$, and deduce the Tate-Nakayama theorem.

b) Explain how the case $n = -2$, $G = \mathrm{Gal}(L/K)$, $A = L^*$ of the Tate-Nakayama theorem gives rise to the local reciprocity map

$$\omega_{L/K}\colon K^*/N_{L/K}(L^*) \xrightarrow{\sim} G^{\mathrm{ab}}.$$