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Exercise Sheet 2

1. Consider the compositum E of all quadratic extensions of $\mathbb{Q}$ inside a fixed algebraic closure $\overline{\mathbb{Q}}$.

a) Show that $\mathrm{Gal}(E/\mathbb{Q})$ is uncountable and has uncountably many subgroups of index 2.

b) Deduce that there are uncountably many subgroups of index 2 in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ that are not open.

2. Let G be a group and X a topological space. An action of G on X is *continuous* if the map $G \times X \longrightarrow X$ is continuous. Prove: if X is a discrete topological space, then an action of G on X is continuous if and only if $\mathrm{Stab}_G(x)$ is open in G for each $x \in X$.

3. Let G be a finite group of automorphisms of a field K and let $k = K^G$ be the fixed field. Prove that K/k is a finite Galois extension with Galois group isomorphic to G.

4. The aim of this exercise is to prove that each profinite group is a Galois group[1].

a) Let K be a field and G a profinite group. Assume that G acts faithfully as a group of automorphisms of K, and that the stabilizer of each element of K is an open subgroup of G. Set $k = K^G$. Prove that K/k is a Galois extension and that the action $G \longrightarrow \mathrm{Aut}(K)$ induces an isomorphism $G \cong \mathrm{Gal}(K/k)$. [Take finitely many elements $x_1, \ldots, x_r \in K$ and look at the extension $k(Gx_1, \ldots, Gx_r)/k$. Find its Galois group in terms of stabilizers. Obtain a map $G \longrightarrow \mathrm{Gal}(K/k)$.]

b) Let G be a profinite group. For each open normal subgroup N of G and each class $c_N \in G/N$, let X_{c_N} be a variable. Fix a field F, and let K be the purely transcendental extension of F generated by all variables X_{c_N}. Define a natural action of G on K satisfying the hypothesis of the previous part and conclude that $G \cong \mathrm{Gal}(K/K^G)$.

[1] This was first proved by Waterhouse in [1]. Note however that one cannot take the extension to be of the form k^s/k. For instance, a celebrated theorem of Artin-Schreier states that a non-trivial finite group G is an absolute Galois group if and only if $G \cong \mathbb{Z}/2\mathbb{Z}$.

Please turn over!

5. Let k be an algebraically closed field of characteristic 0, and $k((t))$ the field of Laurent series on k. The aim of this exercise is to prove that the absolute Galois group of $k((t))$ is isomorphic to $\hat{\mathbb{Z}}$.

a) Let $L/k((t))$ be a finite extension of degree n. Show that there exists $\alpha \in L$ and a polynomial $f(X) \in k[[t]][X]$ monic of degree n in X such that $L = k((t))(\alpha)$, $f(\alpha) = 0$ and $f'(\alpha) \neq 0$.

b) By the implicit function theorem, one can write $t = b_0 + b_1\alpha + b_2\alpha^2 + \cdots \in k[[\alpha]]$. Show that $b_j = 0$ for $j < n$.

c) Prove that there exists $\tau = c_1\alpha + c_2\alpha^2 + \cdots \in k[[\alpha]]$ such that $\tau^n = t$, and that $\alpha \in k[[\tau]]$.

d) Deduce that L embeds into a cyclic degree n extension of $k((t))$ and conclude.

References

[1] W. C. Waterhouse: *Profinite groups are Galois groups.* Proceedings of the American Mathematical Society, Volume 42, Number 2, 1974.