Exercise Sheet 3

1. Let A be a discrete valuation ring and $K = \mathrm{Frac}(A)$. Let $\mathfrak{m}$ be the unique maximal ideal of A and k the residue field of A. Do K and k have the same characteristic?

2. Let K be the fraction field of a discrete valuation ring, with valuation $v : K^\times \longrightarrow \mathbb{Z}$. Fix a real number $0 < a < 1$ and consider the absolute value

$$|x| = a^{v(x)}.$$

Prove that the field K is locally compact with respect to the topology induced by $|\cdot|$ if and only if it is complete and has finite residue field.

3. Let p be a prime number. For each system of "digits" $\alpha_i \in \{0, 1, \ldots, p-1\}$ with $\alpha_i = 0$ for $i \ll 0$, we associate the p-adic number

$$\sum_{i\in\mathbb{Z}} \alpha_i p^i \in \mathbb{Q}_p. \tag{1}$$

a) Explain how each element of $\mathbb{Q}_p$ has a unique expansion as above.

b) We say that the p-adic number $\alpha = \sum_{i\in\mathbb{Z}} \alpha_i p^i$ is *definitely periodic* if there exists a positive integer d such that $\alpha_{i+d} = \alpha_i$ for $i \gg 0$. Prove that the *definitely periodic* p-adic numbers are precisely the rational numbers. [*Hint:* First, consider periodic p-adic integers]

4. For $p \in \{2, 3, 5, 7\}$, decide if the polynomial $X^2 + X + 1$ has a root in $\mathbb{Q}_p$.

5. Let p be a prime number. Show that $\mathbb{Q}_p(\sqrt{p})$ is totally ramified, and it is tamely ramified if and only if $p \neq 2$.

6. Let d be a square-free non-zero integer and p an odd prime number.

a) Show that $\mathbb{Q}_p(\sqrt{d})$ is an unramified extension of $\mathbb{Q}_p$ if and only if $p \nmid d$.

b) Given d, d' squarefree integers with $p \nmid d, d'$, can you find an isomorphism between the corresponding $\mathbb{Q}_p(\sqrt{d}) \cong \mathbb{Q}_p(\sqrt{d'})$ over $\mathbb{Q}_p$?

c) Show that d is a square in $\mathbb{Q}_2$ if and only if $d \equiv 1 \pmod 8$ [*Hint:* There exists $F \in \mathbb{Q}[\![X]\!]$ such that $F^2 = 1 + X$]

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d) Prove that $\mathbb{Q}_2(\sqrt{d})$ is an unramified extension of $\mathbb{Q}_2$ if and only if $d \equiv 1 \pmod 4$.

e) Let $p_1, \dots, p_r$ be distinct prime numbers with $p_j \equiv 1 \pmod 4$ and ℓ any prime number. Show that the ramification index of the extension $\mathbb{Q}_\ell(\sqrt{-p_1 \cdots p_r}, \sqrt{p_1})/\mathbb{Q}_\ell$ is at most 2.

7. Let A be a discrete valuation ring with maximal ideal $\mathfrak{m}$ and $K = \mathrm{Frac}(A)$. Let $P \in A[X]$ be *Eisenstein polynomial*, that is, a polynomial of the form

$$P = X^n + a_{n-1}X^{n-1} + \cdots + a_0$$

with $a_i \in \mathfrak{m}$ for each $i \in \{0, \dots, n-1\}$ and $a_0 \notin \mathfrak{m}^2$. Such a polynomial is irreducible. Assume that A is complete. Prove that the extension $L = K[X]/(P)$ of K is totally ramified, and that all totally ramified extensions of K are obtained in this way.

8. Let k be a field. Show that all finite unramified field extensions of $k((t))$ are of the form $k'((t))$ for k'/k a finite separable field extension.

9. Let k be a perfect field.

a) Prove that $k((t))^{\mathrm{ur}}$ is the union of the $k'((t))$ for k' finite extension of k.

b) Explain why this does **not** imply that $k((t))^{\mathrm{ur}} = \overline{k}((t))$ and try to find an example in which $\overline{k}((t))$ is strictly bigger than $k((t))^{\mathrm{ur}}$.

c) Suppose that k is algebraically closed. Prove that all extensions of $k((t))$ are totally ramified.

10. Let K be a local field with residue field k.

a) Suppose that $K = k((t))$. Choose an infinite set $\{a_0, \dots, a_n, \dots\}$ of elements of $\overline{k}$. Show that $(\sum_{m=0}^{n} a_m t^m)_n$ is a sequence of elements of K^{ur} which does not converge in K^{ur}. Deduce that K^{ur} is not complete with respect to the discrete valuation extending the one on K.

b) What is the completion of K^{ur} for $K = k((t))$?

c) Suppose now that $K = \mathbb{Q}_p$. Show that $\mathbb{Q}_p^{\mathrm{ur}}$ is not complete.

d) Let K be a local field. Endow K^{sep} with the (non-discrete) valuation $v : K^{\mathrm{sep}} \longrightarrow \mathbb{Q} \cup \{\infty\}$ obtained by passing to the limit on the separable finite extensions L of K (where each L is endowed with the valuation $v_L : L \longrightarrow \frac{1}{e_L}\mathbb{Z} \cup \{\infty\}$ extending v, where e_L is the ramification index of L/K). Show that K^{sep} is not complete with respect to this valuation [*Hint:* Observe that the limit of a sequence in K^{ur} converging in K^{sep} lies on a finite extension L of K. Deduce that such a limit must stay in K^{ur}]