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Exercise Sheet 5

1. Let K be a complete valued field with non-discrete topology and let E be a Hausdorff topological vector space over K of finite dimension n. Prove that $E \cong K^n$ as topological vector spaces. [*Hint:* Induction on n]

2. Let A be a DVR with residue field k and let $f \in A[X]$ be an irreducible polynomial. Let $\bar{f} \in k[X]$ be the reduction of f modulo $\mathfrak{m}_A$ and suppose that the extension $k' := k[X]/(\bar{f})$ is a separable field extension of k. Prove that the ring $A[X]/(f)$ is a DVR with residue field k'.

3. **a)** Give a statement of the quadratic reciprocity law.

b) Find for which primes p the polynomial $f(X) = X^2 - 11$ has a root in $\mathbb{Q}_p$.

c) Let now $p = 5$. For each root α of f, compute the coefficients $a_i \in \{0, 1, \dots, 4\}$ of the expansion $\alpha = \sum_{i \in \mathbb{Z}} a_i \, 5^i$ for $i < 5$.

4. Let K be a complete discrete value field and L/K a finite algebraic extension. Let $T \subseteq L$ be the maximal unramified extension of K in L. Let $k \subseteq t \subseteq \ell$ be the associated residue field extensions. Prove: t is the maximal separable extension of k in ℓ.