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Exercise Sheet 6

Let G be a group and A a G-module. Given the abelian groups $K^i = \{f : G^i \longrightarrow A\}$, in class we defined the differentials $d^i : K^i \longrightarrow K^{i+1}$. Let $Z^i(G, A) := \ker d^i$ be the space of *cocycles* and $B^i(G, A) := \operatorname{im} d^{i-1}$ be the space of *coboundaries*. By the theorem seen in class, we have

$$H^i(G, A) \cong Z^i(G, A)/B^i(G, A). \tag{1}$$

Notice that, for $i = 1$, we have

$$Z^1(G, A) = \{f : G \longrightarrow A \,|\, \forall g_1, g_2 \in G, f(g_1g_2) = f(g_1) + g_1 \cdot f(g_2)\}$$

and

$$B^1(G, A) = \{f : G \longrightarrow A \,|\, \exists a \in A : f(g) = g \cdot a - a\}.$$

1. Let $m > 1$ be an integer and $G = \mathbb{Z}/m\mathbb{Z}$.

a) How many G-module structures are there on the abelian group $\mathbb{Z}/12\mathbb{Z}$?

b) Set now $m = 6$ and choose a non-trivial G-module structure on $\mathbb{Z}/12\mathbb{Z}$. Find A^G and $H^1(G, A)$.

2. Let G be a group and let $J(G)$ be the *augmentation ideal of* $\mathbb{Z}[G]$, that is, the kernel of the ring homomorphism $\mathbb{Z}[G] \longrightarrow \mathbb{Z}$ sending all $g \in G$ to 1. Let A be a G-module.

a) Give an isomorphism $Z^1(G, A) \longrightarrow \operatorname{Hom}_G(J(G), A)$.

b) Can you describe the image of $B^1(G, A)$ via this isomorphism?

3. Let G be a finite cyclic group generated by σ and let A be a G-module. Consider the norm map $N : A \longrightarrow A$ given by $a \mapsto \sum_{\tau \in G} \tau a$ and the map $f : A \longrightarrow A$ given by $a \mapsto \sigma a - a$. Show that $H^1(G, A) \cong \ker N/\operatorname{im} f$.

4. Let G be a finite group of order n and let A be a G-module.

a) For $c \in Z^1(G, A)$, let $a = \sum_{g \in G} c(g) \in A$. Show that $(g - 1)a = -nc(g)$. Deduce that $H^1(G, A)$ is annihilated by n.

Please turn over!

b) Suppose furthermore that A is finitely generated as an abelian group. Show that $H^1(G, A)$ is finite. [*Hint:* First, show that the abelian group K^1 of maps $G \longrightarrow A$ is finitely generated].

5. Let I be an injective G-module. Prove that I is a divisible abelian group.