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Exercise Sheet 7

1. Let G be a finite group and let L_i be the free abelian group on G^{i+1}. For $i \geq 1$ consider the differentials

$$\begin{aligned} d_i : L_i &\longrightarrow L_{i-1} \\ (g_0, \ldots, g_i) &\longmapsto \sum_{j=0}^{i} (-1)^j (g_0, \ldots, g_{j-1}, g_{j+1}, \ldots, g_i). \end{aligned}$$

a) Prove that $d_i \circ d_{i+1} = 0$ for each $i \geq 1$.

b) For $i \geq 0$, define the morphisms of abelian groups $u_i : L_i \longrightarrow L_{i+1}$ sending $1 \mapsto 1$ and $(g_0, \ldots, g_i) \mapsto (1, g_0, \ldots, g_i)$. Prove that for each $i \geq 1$ the following equality holds:

$$u_{i-1} \circ d_i + d_{i+1} \circ u_i = \mathrm{id}_{L_i}$$

2. Recall the identification $\mathrm{Hom}_G(L_i, A) \cong \mathrm{Maps}(G^i, A) =: K^i$ seen in class. Prove that the differentials d_i from the previous exercise induce via this identification the differentials $d^i : K^i \longrightarrow K^{i+1}$ defined by

$$\begin{aligned} (d^i f)(g_1, \ldots, g_{i+1}) = g_1 \cdot f(g_2, \ldots, g_{i+1}) &+ \sum_{j=1}^{i} (-1)^j f(g_1, \ldots, g_{j-1}, g_j g_{j+1}, g_{j+2}, \ldots, g_{i+1}) \\ &+ (-1)^{i+1} f(g_1, \ldots, g_i). \end{aligned}$$

3. Write down explicitly what 2-cocycles and 2-coboundaries of a G-module are.

4. Let E be a field and let $\sigma_1, \ldots, \sigma_n$ be distinct automorphisms of E. Prove that if $c_1, \ldots, c_n \in E$ and

$$c_1 \sigma_1(x) + \ldots + c_n \sigma_n(x) = 0$$

for all $x \in E$, then $c_1 = \cdots = c_n = 0$.

5. Let G be a finite group and A a G-module.

a) Prove that the map $A \longrightarrow I_G(A)$ sending a to the map $g \mapsto g \cdot a$ is an injective morphism of G-modules which makes A, as an abelian group, a direct summand of $I_G(A)$.

Please turn over!

b) Deduce that all injective G-modules are relatively injective.

6. Let G be a finite group and let $0 \longrightarrow A \longrightarrow I \longrightarrow B \longrightarrow 0$ be a short exact sequence of G-modules such that I is relatively injective. Prove that $H^i(G,B) \cong H^{i+1}(G,A)$ for all $i > 0$ and that there is a canonical surjection $H^0(G,B) \longrightarrow H^1(G,A)$.

7. Let G be a group and $H \leq G$.

a) Let B be a G-module and A an H-module. Prove that the map

$$\begin{aligned} \vartheta_{B,A} : \mathrm{Hom}_H(B,A) &\longrightarrow \mathrm{Hom}_G(B, I_G^H(A)) \\ \psi &\longmapsto (b \mapsto (g \mapsto \psi(gb))) \end{aligned}$$

defines an isomorphism of abelian groups.

b) What does it mean that $\vartheta_{B,A}$ is natural in A and B? Prove it. One can then say that the functor $I_G^H : H\text{-Mod} \longrightarrow G\text{-Mod}$ is a right adjoint of the forgetful functor $G\text{-Mod} \longrightarrow H\text{-Mod}$.

c) Deduce: if I is an injective H-module, then I_G^H is an injective G-module.

d) What happens when $H = \{1\}$?

8. Let G be a finite group and $H \leq G$.

a) Prove that $\mathbb{Z}[G]$ is a free $\mathbb{Z}[H]$-module.

b) Deduce that any induced G-module is an induced H-module. [*Hint:* Recall that $I_G(B)$ is isomorphic, non-canonically, to $\mathbb{Z}[G] \otimes_{\mathbb{Z}} B$ for each abelian group B]

c) Let A be an H-module. Check that the morphism $I_G^H(A) \longrightarrow A$ sending $u \mapsto u(1)$ is compatible with the inclusion $H \hookrightarrow G$. Recall how this induces natural morphisms

$$H^i(G, I_G^H(A)) \longrightarrow H^i(H,A). \qquad (1)$$

d) (*) (Shapiro's Lemma) Prove that the morphisms (1) are isomorphisms.

9. Let G be a finite group and $H \leq G$. Let A be a G-module. Prove that the restriction-inflation sequence

$$0 \longrightarrow H^1(G/H, A^H) \xrightarrow{\mathrm{Inf}} H^1(G,A) \xrightarrow{\mathrm{Res}} H^1(H,A)$$

is exact. You may use the description in terms of cocycles of the involved maps [*Hint*: Given $f \in \ker(\mathrm{Res})$, then, up to a coboundary, f is trivial on H]