Exercise Sheet 8

In all the exercises, $G$ is a finite group. Given two $G$-modules $X$ and $A$, we endow $\text{Hom}_\mathbb{Z}(X, A)$ and $X \otimes \mathbb{Z} A$ with the $G$-actions defined for all $g \in G$ and $x \in X$ by

$$\forall f \in \text{Hom}_\mathbb{Z}(X, A), \quad (g \cdot f)(x) := g \cdot f(g^{-1}x)$$

$$\forall a \in A, \quad g \cdot (x \otimes a) := (g \cdot x) \otimes (g \cdot a).$$

Notice: if $A$ is an abelian group, we can endow it with trivial $G$-action. In this case we obtain $G$-actions defined by $(g \cdot f)(x) := f(g^{-1}x)$ and $g \cdot (x \otimes a) = (g \cdot x) \otimes a$.

1. (Computation of group homology)
   a) Prove that the functors $(-)_G$ and $\mathbb{Z} \otimes \mathbb{Z}[G]$ −, from the category of $G$-modules to the category of abelian groups, are isomorphic. [Hint: the $\mathbb{Z}[G]$-module structure on $\mathbb{Z}$ is given via the augmentation map]

   Let $\ldots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \mathbb{Z} \rightarrow 0$ be a projective resolution of the trivial $G$-module $\mathbb{Z}$, shortly written as $P_\bullet \xrightarrow{d_0} \mathbb{Z} \rightarrow 0$.

   b) Check that $P_\bullet$ is also a projective resolution of right $G$-modules, the right action of $G$ on each $P_n$ given by $p \cdot g = g^{-1} \cdot p$ for $g \in G$ and $p \in P_n$.

   c) Deduce that $H_n(G, A) = H^n(P_\bullet \otimes \mathbb{Z}[G], A)$. [You may use the fact that, given a ring $R$, a right $R$-module $M$ and a left $R$-module $N$, then $\text{Tor}_i^R(M, N)$ can be equivalently defined as the the $i$-th left derived functor of $- \otimes_R N$ or of $M \otimes_R -$]

2. Prove that relatively injective $G$-modules have trivial homology in positive degree.
   [Hint: Apply the previous exercise on an induced $G$-module]

3. Let $X$ be a $G$-module which is free as $\mathbb{Z}$-module and $A$ any $G$-module.
   a) Prove that the following map is an isomorphism of $G$-modules:

   $$v : X \otimes \mathbb{Z} A \rightarrow \text{Hom}_\mathbb{Z}(\text{Hom}_\mathbb{Z}(X, \mathbb{Z}), A)$$

   $$x \otimes a \mapsto (f \mapsto f(x) a)$$

   b) Endow $X$ with the right $G$-action as in Exercise 1b). Prove that there is a canonical isomorphism $X \otimes \mathbb{Z}[G] A \cong (X \otimes \mathbb{Z} A)_G$.

Please turn over!
4. \textit{(Computation of modified Tate cohomology)} Let $P_\bullet \xrightarrow{\alpha} \mathbb{Z} \rightarrow 0$ be a projective resolution by $G$-modules of $\mathbb{Z}$ (with trivial action), with differentials $d_i : P_i \rightarrow P_{i-1}$ for $i \geq 1$. Consider the $\mathbb{Z}$-dual complex $0 \rightarrow \mathbb{Z} \xrightarrow{\alpha^*} P_\bullet^\ast \rightarrow P_{\bullet}^1 \rightarrow \ldots$ (1) with $P_{n}^\ast := \text{Hom}_{\mathbb{Z}}(P_n, \mathbb{Z})$ and differentials $d_i^\ast$ given by pre-composition with $d_i$.

a) Prove that the complex

\[ \ldots \rightarrow P_1 \rightarrow P_0 \xrightarrow{\alpha^* \circ \alpha} P_0^0 \rightarrow P_1^1 \rightarrow \ldots \] (1)

is exact.

Rename the chain complex (1) as $Q_\bullet$ via $Q_i = P_i$ for $i \geq 0$ and $Q_i = P_{i-1}^\ast$ for $i < 0$.

We want to prove that $\hat{H}^i(G, A) \cong H^i(\text{Hom}_{\mathbb{Z}[G]}(Q_\bullet, A))$ (2) for all $i \in \mathbb{Z}$, that is, that the cochain complex $(\text{Hom}_{\mathbb{Z}[G]}(Q_i, A))_{i \in \mathbb{Z}}$ computes the modified Tate cohomology of $A$.

b) Recall why this is the case for $i \geq 1$.

c) Define natural isomorphisms $\chi_i : P_i \otimes_{\mathbb{Z}[G]} A \rightarrow \text{Hom}_{\mathbb{Z}[G]}(P_i^\ast, A)$ for $i \geq 0$ \textit{[Hint: Use Exercise 3. You may want to use as well the norm map and the fact that induced $G$-modules have trivial Tate cohomology in each degree.]}

d) Prove that the isomorphisms from the previous part respect the differentials and deduce that (2) holds for $i \leq -2$.

e) (*) Conclude by proving (2) in the cases $i = 0$ and $i = -1$. \textit{[Hint: Check, using maps induced by $\alpha$ and $\alpha^\ast$, that the map $\tau$ in $\text{Hom}_{\mathbb{Z}[G]}(Q_{-1}, A) \cong P_0 \otimes_{\mathbb{Z}[G]} A \xrightarrow{\tau} \text{Hom}_{\mathbb{Z}[G]}(P_0, A) = \text{Hom}_{\mathbb{Z}[G]}(Q_0, A)$ factors through the norm map $A_G \rightarrow A_G^G$]}

5. Suppose that $G$ is a finite cyclic group and let $A$ be a $G$-module. If $\hat{H}^0(G, A)$ and $\hat{H}^1(G, A)$ are finite, we define the \textit{Herbrand quotient} of $A$ as

\[ h(A) := \frac{|\hat{H}^0(G, A)|}{|\hat{H}^1(G, A)|}. \]

Assume known the fact that the modified group homology of a finite cyclic group is 2-periodic. Prove the following properties of the Herbrand quotient:

a) Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a short exact sequence of $G$-modules. If two of the Herbrand quotients $h(A)$, $h(B)$ and $h(C)$ are defined, then so is the third and $h(B) = h(A) \cdot h(C)$.

b) If $A$ is a finite $G$-module, then $h(A) = 1$.

c) Let $f : A \rightarrow B$ be a morphism of $G$-modules with finite kernel and cokernel. If one of the Herbrand quotients $h(A)$ and $h(B)$ is defined, then so is the other and they coincide.