Exercise Sheet 8

In all the exercises, G is a finite group. Given two G-modules X and A, we endow $\mathrm{Hom}_{\mathbb{Z}}(X, A)$ and $X \otimes_{\mathbb{Z}} A$ with the G-actions defined for all $g \in G$ and $x \in X$ by

$$\forall f \in \mathrm{Hom}_{\mathbb{Z}}(X, A), \ \ (g \cdot f)(x) := g \cdot f(g^{-1}x)$$
$$\forall a \in A, \ \ g \cdot (x \otimes a) := (g \cdot x) \otimes (g \cdot a).$$

Notice: if A is an abelian group, we can endow it with trivial G-action. In this case we obtain G-actions defined by $(g \cdot f)(x) := f(g^{-1}x)$ and $g \cdot (x \otimes a) = (g \cdot x) \otimes a$.

1. (*Computation of group homology*)

a) Prove that the functors $(-)_G$ and $\mathbb{Z} \otimes_{\mathbb{Z}[G]} -$, from the category of G-modules to the category of abelian groups, are isomorphic. [*Hint:* the $\mathbb{Z}[G]$-module structure on $\mathbb{Z}$ is given via the augmentation map]

Let $\ldots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \mathbb{Z} \longrightarrow 0$ be a projective resolution of the trivial G-module $\mathbb{Z}$, shortly written as $P_\bullet \xrightarrow{d_0} \mathbb{Z} \longrightarrow 0$.

b) Check that $P_\bullet$ is also a projective resolution of right G-modules, the right action of G on each P_n given by $p * g = g^{-1} \cdot p$ for $g \in G$ and $p \in P_n$.

c) Deduce that $H_n(G, A) = H^n(P_\bullet \otimes_{\mathbb{Z}[G]} A)$. [You may use the fact that, given a ring R, a right R-module M and a left R-module N, then $\mathrm{Tor}_i^R(M, N)$ can be equivalently defined as the the i-th left derived functor of $- \otimes_R N$ or of $M \otimes_R -$]

2. Prove that relatively injective G-modules have trivial homology in positive degree. [*Hint:* Apply the previous exercise on an induced G-module]

3. Let X be a G-module which is free as $\mathbb{Z}$-module and A any G-module.

a) Prove that the following map is an isomorphism of G-modules:

$$v : X \otimes_{\mathbb{Z}} A \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}), A)$$
$$x \otimes a \longmapsto (f \mapsto f(x)\, a)$$

b) Endow X with the right G-action as in Exercise 1b). Prove that there is a canonical isomorphism $X \otimes_{\mathbb{Z}[G]} A \cong (X \otimes_{\mathbb{Z}} A)_G$.

Please turn over!

4. (*Computation of modified Tate cohomology*) Let $P_\bullet \xrightarrow{\alpha} \mathbb{Z} \longrightarrow 0$ be a projective resolution by G-modules of $\mathbb{Z}$ (with trivial action), with differentials $d_i : P_i \longrightarrow P_{i-1}$ for $i \geq 1$. Consider the $\mathbb{Z}$-dual complex $0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha_*} P_*^\bullet$ with $P_*^n := \mathrm{Hom}_{\mathbb{Z}}(P_n, \mathbb{Z})$ and differentials d_*^i given by pre-composition with d_i.

a) Prove that the complex

$$\ldots \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\alpha_* \circ \alpha} P_*^0 \longrightarrow P_*^1 \longrightarrow \ldots \tag{1}$$

is exact.

Rename the chain complex (1) as $Q_\bullet$ via $Q_i = P_i$ for $i \geq 0$ and $Q_i = P_*^{-i-1}$ for $i < 0$. We want to prove that

$$\hat{H}^i(G, A) \cong H^i(\mathrm{Hom}_{\mathbb{Z}[G]}(Q_\bullet, A)) \tag{2}$$

for all $i \in \mathbb{Z}$, that is, that the cochain complex $(\mathrm{Hom}_{\mathbb{Z}[G]}(Q_i, A))_{i \in \mathbb{Z}}$ computes the modified Tate cohomology of A.

b) Recall why this is the case for $i \geq 1$.

c) Define natural isomorphisms $\chi_i : P_i \otimes_{\mathbb{Z}[G]} A \longrightarrow \mathrm{Hom}_{\mathbb{Z}[G]}(P_*^i, A)$ for $i \geq 0$ [*Hint:* Use Exercise 3. You may want to use as well the norm map and the fact that induced G-modules have trivial Tate cohomology in each degree.]

d) Prove that the isomorphisms from the previous part respect the differentials and deduce that (2) holds for $i \leq -2$.

e) (*) Conclude by proving (2) in the cases $i = 0$ and $i = -1$. [*Hint:* Check, using maps induced by α and α_*, that the map τ in $\mathrm{Hom}_{\mathbb{Z}[G]}(Q_{-1}, A) \cong P_0 \otimes_{\mathbb{Z}[G]} A \xrightarrow{\tau} \mathrm{Hom}_{\mathbb{Z}[G]}(P_0, A) = \mathrm{Hom}_{\mathbb{Z}[G]}(Q_0, A)$ factors through the norm map $A_G \longrightarrow A^G$]

5. Suppose that G is a finite **cyclic** group and let A be a G-module. If $\hat{H}^0(G, A)$ and $\hat{H}^1(G, A)$ are finite, we define the *Herbrand quotient* of A as

$$h(A) := \frac{|\hat{H}^0(G, A)|}{|\hat{H}^1(G, A)|}.$$

Assume known the fact that the modified group homology of a finite cyclic group is 2-periodic. Prove the following properties of the Herbrand quotient:

a) Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence of G-modules. If two of the Herbrand quotients $h(A)$, $h(B)$ and $h(C)$ are defined, then so is the third and $h(B) = h(A)\,h(C)$.

b) If A is a finite G-module, then $h(A) = 1$.

c) Let $f : A \longrightarrow B$ be a morphism of G-modules with finite kernel and cokernel. If one of the Herbrand quotients $h(A)$ and $h(B)$ is defined, then so is the other and they coincide.