Exercise Sheet 9

1. Let p be a prime number, G a p-group and A a non-zero G-module. Assume that every element of A has order a power of p. Show that $A^G \neq \{0\}$.

2. Let G be a finite group and A and B two G-modules. Prove that, if B is induced, then the G-module $\mathrm{Hom}(A, B)$ is induced as well. (Recall that a G-module B is induced if one can find a subgroup X of B such that $B = \bigoplus_{g \in G} gX$).

3. The goal of this exercise is to establish the following:

Theorem 1. *Let p be a prime number, G a p-group, and A a p-torsion G-module such that $\widehat{H}^n(G, A) = 0$ for some integer n. Then A is an induced G-module.*

The proof will be divided into several steps:

a) Let $\Lambda = \mathbb{F}_p[G]$. Choose a basis I of the $\mathbb{F}_p$-vector space A^G and set $V = \bigoplus_I \Lambda$. Observe that there is an isomorphism $j_G \colon A^G \simeq V^G$.

b) Deduce from the long exact sequence associated to

$$0 \to \mathrm{Hom}(A/A^G, V) \to \mathrm{Hom}(A, V) \to \mathrm{Hom}(A^G, V) \to 0$$

and Exercise 2 that there is a surjective map

$$\mathrm{Hom}_G(A, V) \longrightarrow \mathrm{Hom}(A^G, V^G).$$

It follows that j_G can be extended to a G-morphism $j \colon A \to V$.

c) Prove that j is injective (*Hint*: use Exercise 1 for the G-module $\ker j$).

d) Prove that $H^1(G, A) = 0$ implies that j is surjective, hence that $A \simeq V$ is an induced G-module.

e) Recall the following form of the "shift trick": given a G-module A, we define a sequence of G-modules $(A_r)_{r \in \mathbb{Z}}$ as follows:

- $A_0 = A$
- A_1 if defined by the exact sequence $0 \to A \to I_G(A) \to A_1 \to 0$; for each $q \geq 2$, we recursively define $A_q = (A_{q-1})_1$
- A_{-1} is defined by the exact sequence $0 \to A_{-1} \to I_G(A) \to A \to 0$ and, for each $q \leq -2$, we recursively define $A_q = (A_{q+1})_{-1}$.

Please turn over!

For all integers q, r, there is an isomorphism:

$$\widehat{H}^q(G, A) \simeq \widehat{H}^{q-r}(G, A_r).$$

f) Use part **d)** and the shift trick to prove that the assumption $\widehat{H}^n(G, A) = 0$ for some integer n implies that $H^1(G, A) = 0$ and therefore that A is induced. This concludes the proof.